STUDY OF ROBUSTNESS OF OPTIMAL DIGITAL CONTROLLER WITH A PRESCRIBED DEGREE OF STABILITY

A Thesis Submitted

in partial Fulfillment of the Requirements

for the Degree of

Master of Technology

by

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to the

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

June, 1992

CERTIFICATE

This is to certify that this work in the thesis entitled "Study Of Robustness Of Optimal Digital Controller With A Prescribed Degree Of Stability", by M.V. Tamhankar has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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ABSTRACT

The robustness properties of discrete time linear quadratic regulator are studied. The return difference equality for a discrete time linear quadratic regulator with a prescribed degree of stability is achieved. It is found that the guaranteed stability margins achieved reduce as the degree of stability is increased. It is shown that the stability margins achieved are a measure of nearness of the closed loop poles to a circle which is interior to the unit circle on the z-plane. Systems with lesser degrees of stability, are shown to possess these guaranteed stability margins.

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DEDICATED TO

MY PARENTS

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ACKNOWLEDGEMENTS

I take this opportunity to express my sincere gratitude towards my thesis advisor Prof K E Holé for the valuable guidance and inspiration he has provided me throughout this work as well as during my course work I am greatly indebted to him for the cooperation extended to me without which this thesis would not have taken this form.

I am very grateful to G K. Singh for his valuable assistance which helped me a lot in completing this work.

Thanks are due to Prof R. N. Biswas and Prof. G. C. Ray for their timely help and cooperation. I am thankful to Dr. V. R. Sule for his valuable suggestions on the topic.

My thanks to Medh who has been of great help while preparing the copies of this document.

I am thankful to all my *thulla*–mates – Wadya, Baptya, Nana, Madhu, Baman, Vinu. Bapu, Rahulya, Ghattu, Sagar, Jaina, Rajan, Khad, Mangu, Gogya, Bhandi, Tubale, Pankha, Makya, Kak, Rajesh^2, Karandezvous, Jayant, Sudhir..... and all Hall–4ites who have made my stay *fullto Aish*–ful. Special thanks are also due to the TDP cricket team, especially Ravi, Subba, Vissu et al for the unforgettable matches with us, the All Ghats Aisho–ciation.

Special thanks to Madanji and family who provided me with a home away from home. Shri Ghorpade and Shri Chakradev also deserve thanks.

Manoj

(Tatya)

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LIST OF SYMBOLS

Symbols used frequently are listed below.

A	State matrix for discrete-time systems
B	Input matrix for discrete-time systems
F	State matrix for continuous-time systems
G	Input matrix for continuous-time systems
J	Performance index for DLQR
J_c	Performance index for CLQR
J_α	Performance index for DLQR with prescribed degree of stability
J_ε	Performance index for CLQR with prescribed degree of stability
P	Solution of DARE
P_α	Solution of DARE with prescribed degree of stability
Q	State weighting matrix for DLQR
Q_c	State weighting matrix for CLQR
R	Input weighting matrix for DLQR
R_c	Input weighting matrix for CLQR
α,β	Degree of stability for DLQR
ε	Degree of stability for CLQR
Λ	Lower bound radius on determinant of RDM of DLQR
Λ_α	Lower bound radius on determinant of RDM of DLQR with prescribed degree of stability

LIST OF ABBREVIATIONS

CARE : Continuous-time Algebraic Riccati Equation

CL : Closed Loop

CLQR : Continuous-time Linear Quadratic Regulator

DARE : Discrete-time Algebraic Riccati Equation

DLQR : Discrete-time Linear Quadratic Regulator

LQR : Linear Quadratic Regulator

LTI : Linear Time Invariant

MIMO : Multi Input Multi Output

n.s.d. : negative semi definite

p.d. : positive definite

p.s.d. : positive semi definite

RDM : Return Difference Matrix

SISO : Single Input Single Output

CHAPTER # 1

INTRODUCTION

The thesis attempts robust controller design for a linear time invariant (LTI) discrete time system,

$$x_{k+1} = A\, x_k + B\, u_k$$

using the design model,

$$x_{k+1} = A_o\, x_k + B\, u_k$$

subject to minimization of the quadratic cost function with prescribed degree of stability, $\alpha > 1$, viz.,

$$J = \sum_{k=o}^{\infty} \alpha^{2k} [\, x_k^T\, Q\, x_k + u_k^T\, R\, u_k \,] \quad ;$$

where Q is a positive semi definite (p.s.d.) matrix ($Q \geq 0$) and R is a positive definite (p.d.) matrix ($R > 0$).

The robustness properties of the continuous time linear quadratic regulator (CLQR) are well-known [1-4]. It is shown that for the continuous time LTI system

$$\dot{x} = F\, x + G\, u$$

the controller,

$$u = -\, L\, x$$

which minimizes the quadratic cost,

$$J_c = \int_o^{\infty} [\, x^T Q_c x + u^T R_c u \,]\, dt; \quad Q_c \geq 0 \,, \; R_c > 0$$

gives rise to the closed loop (CL) system

$$\dot{x} = (\, F - G\, L \,)\, x \,.$$

This CL system has excellent robustness properties,

The advantages of CLQR with prescribed degree of stability ε ($\varepsilon > 0$), are brought out by B.D.O.Anderson and J.B.Moore [5]. Thus, instead of using the cost function J_c, above, if one uses the cost function,

$$J_{\varepsilon} = {}_{o}\int^{\infty} e^{2\varepsilon t} [\; x^T Q_C x + u^T R_C u \;] \, dt$$

$$Q_C \geq 0 \; ; \; R_C > 0$$

the optimal CL system has the following advantages:

i) The reduction of trajectory sensitivity to plant parameter variations as a result of any CL control is greater for a regulator with $\varepsilon > 0$ than for $\varepsilon = 0$ (Note that with $\varepsilon = 0$, J_{ε} becomes J).

ii) There is inherently a greater margin for tolerance of time delay in closed loop when $\varepsilon > 0$.

iii) There is greater tolerance to non-linearity when $\varepsilon > 0$.

iv) Asymptotically stable bang-bang control may be achieved simply by inserting a relay in the closed loop when $\varepsilon > 0$.

Thus, in general, the CLQR with $\varepsilon > 0$, possesses better robustness properties. The manifestation of these robustness properties of CLQR with $\varepsilon > 0$ is given in [6]. In that paper, a sufficient condition is derived (on the plant perturbation matrix and ε , the degree of stability), which when satisfied guarantees optimality (and hence the robustness properties) of the nominally optimal control law for perturbed systems, when fixed perturbations occur in plant state matrix. Thus, ε is used as a design parameter to achieve robustness for the perturbed (actual) system. This

It is shown in [7] that the stability margins of the CLQR do not necessarily guarantee the robustness of "cheap" CLQRs which allow large control signals at the input. In [8], it is pointed out that there are essential differences between the CLQR and the discrete time LQR (DLQR). The effect of these differences on the stability margins and the sensitivity properties of DLQR is analyzed. These gain and phase margins for a single input-single output (SISO) DLQR are derived there. It is found that these margins are less than those obtained for CLQR. Moreover, the margins are shown to be dependent upon the performance criterion used. It is also shown that for some frequencies, the closed loop system is more sensitive than the equivalent open-loop system. The reason for this phenomenon is stated to be the inherent delay in the discrete-time systems which has a destabilizing effect.

In [9], these margins for multivariable systems are found out. It is shown there that the stability margins derived for DLQR provide a reliable indication of the robustness of the corresponding DLQR CL system (unlike the case of CLQR, as stated earlier). Thus, although the guaranteed stability margins for the DLQR are small, they are a reliable indication of the robustness of the system.

With this background, a development similar to [5,6] would be useful and this has been the motivation of this work.

Much research is being done in using prescribed degree of stability for robustness improvement in discrete-time systems. In [10], a method is proposed for sensitivity reduced design of

ensures the prescribed degree of stability. In [11], time domain conditions are presented on the appropriate deterministic or random characteristics of perturbations to maintain the proper stability behavior of the overall system. The system is assumed to be nominally stable having certain degree of stability. In [12], the stabilization of systems via LQR is considered giving bounds on the interval system matrices.

This thesis attempts to establish the robustness properties of DLQR with prescribed degree of stability and manifest these in terms of the perturbation in plant state matrix.

Organization Of The Thesis :

Chapter 2 consists of the background material required. The spectral factorisation of CLQR and its robustness properties are briefly reviewed. The detailed development of the spectral factorisation of DLQR and the associated gain and phase margins is given thereafter. The control laws for CLQR and DLQR with prescribed degree of stability are given afterwards.

In Chapter 3 , the design procedure and the condition on the perturbation matrix , as developed in [6] are discussed. Then a similar approach is tried out for DLQR. The return difference equality for DLQR with prescribed degree of stability is derived. Then its associated robustness is discussed. The implications are then confirmed by presenting numerical examples.

In the last chapter , some observations are made and conclusions stated.

CHAPTER # 2

THE LQR AND ITS ROBUSTNESS

2.1 Introduction

It is well-known that the return difference matrix plays an important role in deciding the robustness properties of the system. The minimum, over the entire frequency band, of the smallest singular value of the return difference matrix, provides a measure of the stability margins for the system. The spectral factorisation of the return difference matrix, then, becomes useful in deciding the size of the return difference matrix. In this Chapter, the development of the return difference equality for both, CLQR as well as DLQR, and the corresponding stability margins are reviewed. A brief introduction to the CLQR and DLQR with prescribed degree of stability follows thereafter.

2.2 The Continuous Time LQR

Consider an LTI continuous-time system

$$\dot{x} = F\ x + G\ u \qquad \text{----(2.1)}$$

where x is the n-dimensional state vector ,i.e.$x \in \mathbb{R}^n$.

u is the m-dimensional input vector ,i.e.$u \in \mathbb{R}^m$.

$F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times m}$ are constant state and constant input matrices respectively.

It is required to minimize the cost function

$$J = \int^{\infty} [\ x^T Q\ x + u^T R\ u\ ]\ dt \qquad \text{----(2.2)}$$

Both Q_c and R_c are assumed to be symmetric and real.

It is assumed that (F,G) is controllable and (F,D_c) is observable; where $D_c^T D_c = Q_c$.

The control law which minimizes J_c in (2.2) is given by,

$$u = - L x \qquad \text{----(2.3)}$$

where

$$L = R_c^{-1} G^T M \qquad \text{----(2.4)}$$

$M \in \mathbb{R}^{n\times n}$ is a p.d., symmetric, unique, real solution to the continuous-time algebraic riccati equation (CARE) ,

$$F^T M + M^T F - M G R_c^{-1} G^T M + Q_c = 0 \qquad \text{----(2.5).}$$

From the CARE, the spectral factorisation of the return difference matrix (RDM) of the CL System,

$$\dot{x} = (F - G L) x \qquad \text{----(2.6)}$$

is achieved as follows:

Add and subtract sM from CARE (2.5) to obtain,

$$(-sI - F^T) M + M (sI - F) + L^T R_c L = Q_c$$

Here and hence forth, I indicates an identity matrix of appropriate dimensions. Multiply on the left by $R_c^{-1/2} G^T (-sI - F^T)^{-1}$ and on the right by $(sI - F)^{-1} G R_c^{-1/2}$ to get

$$R_c^{-1/2} G^T (-sI - F^T)^{-1} M G R_c^{-1/2} + R_c^{-1/2} G^T M (sI - F)^{-1} G R_c^{-1/2}$$
$$+ R_c^{-1/2} G^T (-sI - F^T)^{-1} L^T R_c L (sI - F)^{-1} G R_c^{-1/2}$$
$$= R_c^{-1/2} G^T (-sI - F^T)^{-1} Q_c (sI - F)^{-1} G R_c^{-1/2} \text{---(2.7).}$$

Now, since

$$L = R_c^{-1} G^T M$$

we have

$$M\ G\ R_c^{-1/2} = L^T R_c^{1/2} \qquad \text{----(2.9)}.$$

Add I to both sides of (2.7) and use identity (2.9), and replace s by $j\omega$, to get

$$[I+R_c^{1/2}L(-j\omega I-F)^{-1}G\ R_c^{-1/2}]^T\ [I+R_c^{1/2}L(j\omega I-F)^{-1}G\ R_c^{-1/2}]$$
$$= I + R_c^{-1/2}\ G^T(-j\omega I - F^T)^{-1}Q_c(j\omega I - F)^{-1}\ G\ R_c^{-1/2}$$

$$\Rightarrow [I+R_c^{1/2}L\ \Phi_c G\ R_c^{-1/2}]^*\ [I+R_c^{1/2}L\ \Phi_c G\ R_c^{-1/2}]$$
$$= I + R_c^{-1/2}G^T\Phi_c^*\ Q_c\Phi_c G\ R_c^{-1/2} \qquad \text{----(2.10)}$$

$$\Rightarrow \qquad [\ I + L\ \Phi_c G\ ]^*\ R_c\ [\ I + L\ \Phi_c G\ ]$$
$$= R_c + G^T\Phi_c^*\ Q_c\Phi_c G \qquad \text{----(2.10-a)}$$

where $\Phi_c = (j\omega I - F)^{-1}$ and ` * ' denotes conjugate transpose.

Equation (2.10) is the spectral factorisation of the RDM or the so-called return difference equality of the CL system (2.6). Note that (2.10) is valid for all $\omega \in [o,\infty]$.

Now since Q_c is p.s.d., $R_c^{-1/2}G^T\Phi_c^*\ Q_c\Phi_c G\ \ R_c^{-1/2}$ is also p.s.d. for all $\omega \in [o,\infty]$. Hence the following inequality holds.

$$[I+R_c^{1/2}L\ \Phi_c G\ R_c^{-1/2}]^*\ [I+R_c^{1/2}L\ \Phi_c G\ R_c^{-1/2}]\ \geq\ I \qquad \text{----(2.11)}.$$

It is shown in [1], that (2.11) is a necessary and sufficient condition for optimality of the system. The optimality here, is in the sense of the following definition from [1].

<u>Definition</u>: In connection with the LQR problem ,a feedback law is optimal if there exists a p.d. or p.s.d. state weighting matrix and a p.d. input weighting matrix such that the resulting quadratic cost function is minimized by the feedback law under consideration.

For an SISO system, the inequality (2.11) becomes

$$[\ 1 + L\ \Phi_c G\ ]^*\ [\ 1 + L\ \Phi_c G\ ] \geq 1 \ \text{for all}\ \omega \in [o,\infty]$$

This means that for the optimal CL system, the Nyquist plot of the return difference will not enter the unit circle centered at (-1, j0) on the return-difference plane.

This indicates that the optimal CL system possesses infinite gain margin, 50 % gain reduction tolerance and a phase margin of at least ±60 degrees.

For an MIMO system, it is shown in [13] that if N is the perturbation in the system as shown in fig. 2-a ,then the CL system remains stable for N satisfying the following :

$$N = \mathrm{diag}(n_1, n_2, \ldots n_m) \qquad \text{and}$$

$$| n_i^* + n_i | > 1 \qquad \text{----(2.12)}$$

If N is real ,it can be easily shown that (2.12) is satisfied when $n_i \in (1/2, \infty)$. This is the gain margin analogy.

Also if $n_i = e^{-j\psi}$,with $|\psi| < \pi/3$, (2.12) is satisfied. This is the phase margin analogy.

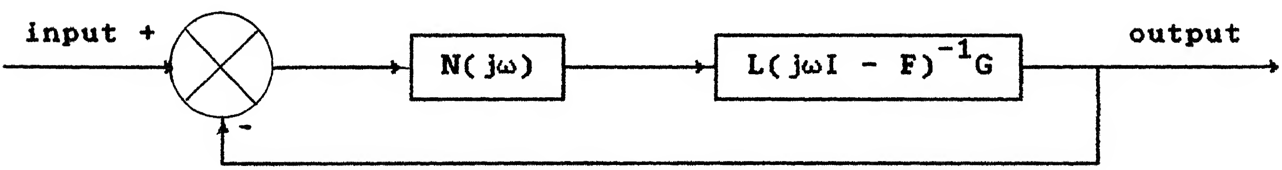


Figure 2-a

Continuous time system with perturbations

2.3 The Discrete Time LQR

Consider a discrete-time LTI system ,whose dynamics is modeled as

$$x_{k+1} = A x_k + B u_k \qquad ----(2.13)$$

where x_k is the state vector at the k^{th} instant, $x_k \in \mathbb{R}^n$

u_k is the input vector at the k^{th}, $u_k \in \mathbb{R}^m$

$A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant state and constant input matrices respectively.

It is required to minimize the performance index,

$$J = \sum_{k=o}^{\infty} [\, x_k^T Q x_k + u_k^T R u_k \,] \quad ; \quad Q \geq 0 \, , \, R > 0 \qquad ----(2.14)$$

where both Q and R are symmetric and real with $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$.

It is assumed that (A,B) is controllable and (A,D) is observable; where $D^T D = Q$.

It is known,[14], that the control law,

$$u_k = - K x_k \qquad ----(2.15)$$

where

$$K = (R + B^T P B)^{-1} B^T P A \qquad ----(2.16)$$

is the optimal control law.

Here, 'P' is the p.d. symmetric unique real solution to the discrete-time ARE (DARE)

$$P - A^T P A + A^T P B K = Q \quad . \qquad ----(2.17).$$

The corresponding CL system is then

$$x_{k+1} = (A - B K) x_k \quad . \qquad ----(2.18).$$

From DARE (2.17), the spectral factorisation of the

Add `$z(A^TP-A^TP) + z^{-1}(P\ A-P\ A)$' to the L.H.S. of (2.17) to get

$$z(A^TP-A^TP) + z^{-1}(P\ A-P\ A)+ P - A^TP\ A + A^TP\ B\ K = Q \quad .$$

Rearranging , we get

$$(z^{-1}I - A^TP)\ P\ (zI - A) + A^TP(zI - A)$$
$$+ (z^{-1}I - A^TP)P\ A + A^TP\ B\ K = Q \quad .$$

Now multiply on the left by $\Phi^* = (z^{-1}I - A^T)^{-1}$ and on the right by $\Phi = (zI - A)^{-1}$, to obtain

$$P + \Phi^*A^TP + P\ A\ \Phi + \Phi^*A^TP\ B\ K\ \Phi = \Phi^*Q\ \Phi \quad .$$

Also multiplying on the left by B^T and on the right by B gives

$$B^TP\ B + B^T\Phi^*A^TP\ B + B^TP\ A\ \Phi\ B + B^T\Phi^*A^TP\ B\ K\ \Phi\ B = B^T\Phi^*Q\ \Phi\ B \ .$$

Add ` R ' to both sides to obtain

$$R + B^TP\ B + B^T\Phi^*A^TP\ B + B^TP\ A\ \Phi\ B + B^T\Phi^*A^TP\ B\ K\ \Phi\ B$$
$$= R + B^T\Phi^*Q\ \Phi\ B$$

i.e.

$$R + B^TP\ B + B^T\Phi^*A^TP\ B(R + B^TP\ B)^{-1}(R + B^TP\ B)$$
$$+(R + B^TP\ B)(R + B^TP\ B)^{-1}B^TP\ A\ \Phi\ B$$
$$+ B^T\Phi^*A^TP\ B(R + B^TP\ B)^{-1}(R + B^TP\ B)K\ \Phi\ B$$
$$= R + W^*Q\ W$$

where $W = \Phi\ B$.

This simplifies to

$$(I + B^T\Phi^*K^T)(R + B^TP\ B)(I + K\ \Phi\ B) = R + W^*Q\ W \qquad \text{----(2.19)}.$$

(2.19) is the return difference equality.

Using (2.19) ,a necessary condition for optimality of (2.18) can be derived as follows [16].

$$F^T(z^{-1})\ (R + B^T P\ B)\ F(z) = R + W^* Q\ W\quad .$$

Taking determinant of both sides of the above equation we have

$$\det[\ F^T(z^{-1})\ F(z)\ ] = \frac{\det[\ R + W^* Q\ W\ ]}{\det[\ R + B^T P\ B\ ]}$$

$$= \frac{\det[\ I + R^{-1/2}\ W^* Q\ W\ R^{-1/2}\ ]}{\det[\ I + R^{-1/2}\ B^T P\ B\ R^{-1/2}\ ]} \qquad \text{----(2.20).}$$

Let

$\gamma_i(z)$ = Eigenvalues of $R^{-1/2} W^* Q\ W\ R^{-1/2}$

β_i = Eigenvalues of $R^{-1/2} B^T P\ B\ R^{-1/2}$

$\rho_i(z)$ = Eigenvalues of $F(z)$

where $i = 1,..,m$.

Now, eigenvalues $\gamma_i(z)$ are non-negative when $|z| = 1$, since Q is p.s.d.,and hence $R^{-1/2} W^* Q\ W\ R^{-1/2}$ is also p.s.d. for $|z| = 1$. Similarly eigenvalues β_i are real, positive since the matrix $R^{-1/2} B^T P\ B\ R^{-1/2}$ is symmetric ,p.d. and constant.

On the other hand,

$$\det[\ F(z)] = \prod_{i=1}^{m} \rho_i(z)$$

and

$$|F^T(z^{-1})\ F(z)| = |F(z)|^2 \qquad \text{for } |z| = 1\quad .$$

Then we have from (2.20)

$$\prod_{i=1}^{m} |\rho_i(z)|^2 = \frac{\prod_{i=1}^{m} [\ 1 + \rho_i\ (z)]}{\prod_{i=1}^{m} [\ 1 + \beta_i\ (z)]} \qquad \text{----(2.21).}$$

For $m = 1$, we have

$$|\rho_i(z)|^2 = \frac{1}{1+\beta_i} + \frac{\gamma_i(z)}{1+\beta_i} .$$

Since $\gamma_i(z)$ is non-negative and β_i is positive,

$$|\rho_i(z)|^2 \geq \frac{1}{1+\beta_i}$$

or,

$$|\rho_i(z)| \geq \frac{1}{(1+\beta_i)^{1/2}} .$$

Similarly for $m = 2$, it can be shown that

$$|\rho_1(z)||\rho_2(z)| \geq \frac{1}{[(1+\beta_1)(1+\beta_2)]^{1/2}} .$$

Proceeding in a similar way, it can be generally written that, for $|z|=1$,

$$|\det[F(z)]| \geq \{ \det[I + R^{-1/2}B^T P B R^{-1/2}] \}^{-1/2} \quad \text{--(2.21)}.$$

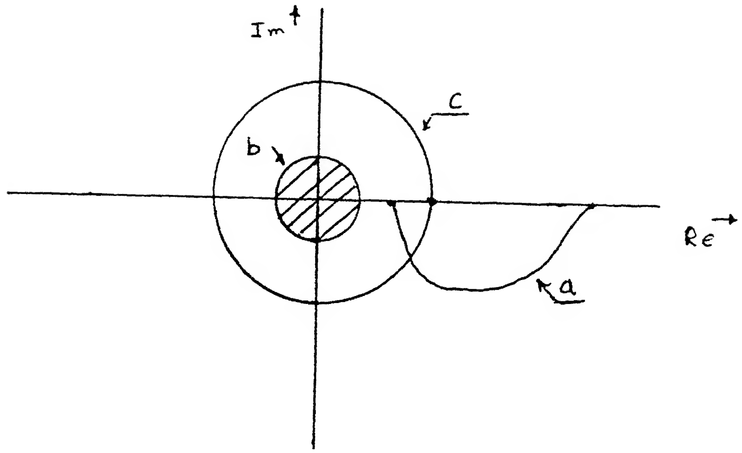
It is then concluded that a necessary condition for optimality of (2.15) subject to minimization of cost J in (2.14) is that the complex plane plot of $|\det[F(z)]|$, when $|z|= 1$,must not penetrate the interior of the circle centered at the origin and having the radius of

$$\Lambda = \{ \det[I + R^{-1/2}B^T P B R^{-1/2}] \}^{-1/2} \quad \text{----(2.23)}$$

Note that Λ is always less than unity.

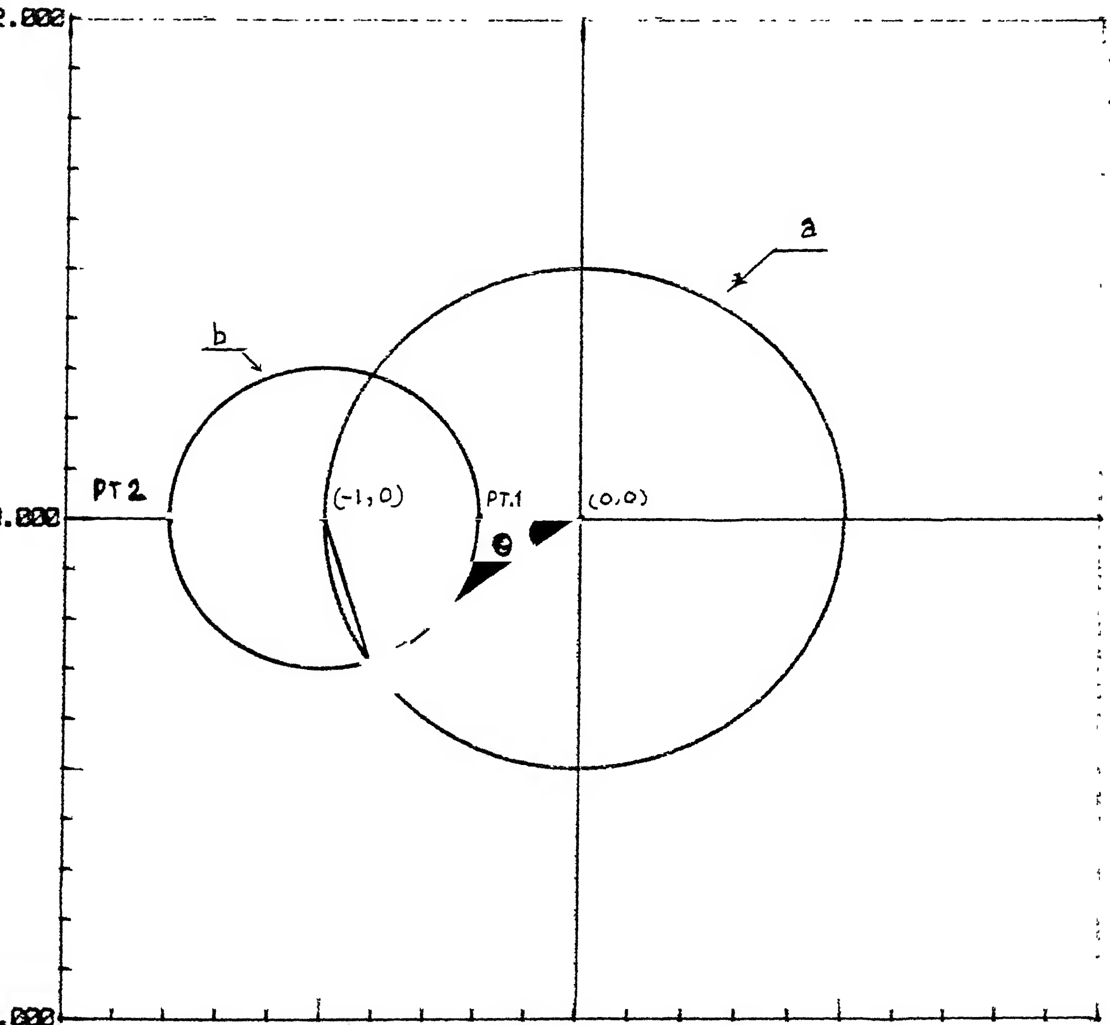
The graphical representation of this optimality property is shown in fig. 2-b.

For an SISO system, (2.23) would mean that the Nyquist plot of the loop transfer function of an optimal system will not penetrate a circle of radius $\Lambda = \{ [1 + b^T P b]/ r\}^{-1/2}$,centered at



a → det (F(Z))
b → forbidden re
optimality
c → unit circle

figure 2-b Graphical representation of optimality condi



_b _circle of radi
_a _unit circle

phase margin. The upward gain margin is calculated at point 1 and the gain reduction tolerance is calculated at point 2.

From the geometry of the figure, it is seen that

$\sin(\theta/2) = \Lambda / 2$.

Hence,

$\theta = 2 \sin^{-1}(\Lambda / 2)$.

It is seen that at point 1, $|F(z)| = 1-\Lambda$. Hence the system would become unstable (i.e. $|F(z)|$ would reach the point (-1,0)), only when it is multiplied by a gain factor of $(1-\Lambda)^{-1}$ or a value less than that. Thus the upward gain margin is $(1-\Lambda)^{-1}$.

Similarly, at point 2, $|F(z)| = (1+\Lambda)$. Hence the downward gain margin is $(1+\Lambda)^{-1}$.

In [9], the gain and phase margins for an MIMO system have been defined and these are obtained for the DLQR. The development goes as follows.

The basic condition for stability of an MIMO ,CL system is that there exists a positive scalar R_f , such that

$\underline{\sigma}(F(z)) \geq R_f > 0$ for all $|z| = 1$.

Here F(z) is the RDM of the system under consideration and $\underline{\sigma}(\cdot)$ denotes the smallest singular value of $(\cdot)$.

The size of the guaranteed stability margins of the DLQR van be related to the value of R_f.

The minimum guaranteed upward and downward gain margins of the LQR are defined as the positive scalars GM_1 and GM_2, for which , a simultaneous insertions of gains g_i ,in the i^{th} feed back loop of the CL regulator [i.e. multiplying all the elements of the

$$GM_2 \le g_i \le GM_1 \quad ; \; i = 1,..,m.$$

Similarly, the guaranteed phase margin of the regulator is defined to be the scalar PM for which a simultaneous insertion of the phase factor $e^{j\theta_i}$ in the i^{th} feed back loop will not destabilize the system if ,

$$| \theta_i | \le PM \quad ; \; i = 1,..,m.$$

The following theorem in [9] gives the guaranteed gain and phase margins for the DLQR in terms of the system elementary matrices and the weighting matrices in the performance index.

<u>Theorem 2.1</u>: The minimum singular value of the RDM F(z), of the DLQR, is bounded from below, for z that traverses the unit circle, by R_f, where,

$$R_f^2 = \frac{\underline{\sigma}(R)}{\bar{\sigma}(R) + \bar{\sigma}^2(B)} \left[1 + \frac{\underline{\sigma}^2(R)\,\underline{\sigma}^2(Q)}{\bar{\sigma}(R)[1 + \bar{\sigma}(A)]^2}\,\chi \right]$$

where, $\chi = 1$ for $m \le n$

$= 0$ otherwise.

$\bar{\sigma}(\cdot)$ indicates the largest singular value of $(\cdot)$.

$$\delta = \bar{\sigma}(Q) \; [\; 1 - \bar{\sigma}^2(A)]^{-1} \qquad \text{for case 1.}$$

$$= \bar{\sigma}(Q) \; [\; 1 - \bar{\alpha}^2]^{-1} \qquad \text{for case 2.}$$

$$= [\bar{\sigma}(Q) + \gamma^2] \; [\; 1 - \hat{\alpha}^2]^{-1} \qquad \text{for case 3.}$$

$$\bar{\alpha} = \max_i | \lambda_i(A) | \quad ; \; i = 1,..,n.$$

$$\gamma = \bar{\sigma}(V_2)\,\bar{\sigma}(V_2 B)\,\underline{\sigma}^{-1}(A_2)\bar{\sigma}(R)(\bar{\alpha}^2 - 1)\underline{\sigma}^{-1/2}(R)\underline{\sigma}^{-2}(B^T V_2)$$

where A_2 and V_2 are defined by the following spectral decomposition of A.

$$A = [\; U_1 \quad U_2 \;] \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad ; \quad [\; U_1 \quad U_2 \;] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = I$$

where A_1 consists of Jordan blocks that correspond to eigenvalues inside the unit circle and A_2 consists of all the other Jordan blocks.

$$\hat{\alpha} = \max_i \{ \alpha' , 1 / \alpha'' \}$$

$$\alpha' = \max_i |\hat{\lambda}_i(A)|$$

$$\alpha'' = \max_i |\hat{\lambda}'_i(A)|$$

$\hat{\lambda}_i(A)$ are the eigen-values of A outside unit circle.

$\hat{\lambda}'_i(A)$ are the eigen-values of A inside unit circle.

The three cases referred to above are the following ones:

Case 1: $\lambda_i(A) < 1$ and $\bar{\sigma}(A) < 1$ for all $i = 1,..,n$.

Case 2: $\lambda_i(A) < 1$ and $\bar{\sigma}(A) \geq 1$ for all $i = 1,..,n$ and all $\lambda_i(A)$ are distinct.

Case 3: $\lambda_i(A)$ are all distinct , $|\lambda_i(A)| \neq 1$, $\lambda_i(A) > 1$ for some i, $\lambda_i(A) \neq \lambda_j^{-1}(A)$, for all $i = 1,..,n$ and for all $j = 1,..,n$.

The guaranteed upward and downward gain margins of the regulator are given by

$$GM_2 = (1 + R_f)^{-1} \quad ; \; GM_1 = (1 - R_f)^{-1} \qquad ----(2.24)$$

and the positive and negative phase margins of this regulator are given by

$$PM = \pm \cos^{-1}[\; 1 - \tfrac{1}{2} R_f^2 \;] \qquad ----(2.25).$$

2.4 The CLQR With Prescribed Degree Of Stability

If the performance index of (2.2) is modified to

$$J_{c\varepsilon} = \int_0^{\infty} e^{2\varepsilon t} [\; x^T Q_c x + u^T R_c u \;] \, dt \qquad ----(2.26)$$

where $Q_c \geq 0$, $R_c > 0$ and ε is a positive scalar, then the optimal control law which minimizes (2.26) subject to

$$u = - L_{\varepsilon} x \qquad \text{----(2.27)}$$

where

$$L_{\varepsilon} = R_C^{-1} G^T M_{\varepsilon} \qquad \text{----(2.28).}$$

$M_{\varepsilon} \in \mathbb{R}^{n\times n}$ is a p.d. symm. unique real solution to the CARE

$$[F + \varepsilon I]^T M_{\varepsilon} + M_{\varepsilon}^T [F + \varepsilon I] - M_{\varepsilon} G R_C^{-1} G^T M_{\varepsilon} + Q_C = 0 \qquad \text{----(2.29).}$$

The resulting CL system is

$$\dot{x} = (F - G L_{\varepsilon}) x \qquad \text{----(2.30)}$$

and this CL system possesses a prescribed degree of stability ε [13]. In other words, the CL poles of the system (2.30) lie on the left half of the vertical line through the point $(-\varepsilon, j0)$, on the s-plane.

2.5 The DLQR With Prescribed Degree Of Stability

In the discrete-time case also, introducing a weighting term into the performance index allows to achieve a prescribed degree of stability [17].

Suppose that we modify the performance index of (2.14) to

$$J_{\alpha} = \sum_{k=0}^{\infty} \alpha^{2k} [x_k^T Q x_k + u_k^T R u_k] \quad ; \quad Q \geq 0 \ , \ R > 0 \ \text{-----(2.31)}$$

where $\alpha > 1$ is a scalar.

We can distribute α^{2k} in J_{α} as $\alpha^k \alpha^k$ and write it as

$$J_{\alpha} = \sum_{k=0}^{\infty} [(\alpha^k x_k^T) Q (\alpha^k x_k) + (\alpha^k u_k^T) R (\alpha^k u_k)]$$

$$= \sum_{k=0}^{\infty} [z_k^T Q z_k + v_k^T R v_k] \qquad \text{----(2.32)}$$

The equations in z and v are readily found. Using (2.33) we have

$$z_{k+1} = \alpha^{k+1} x_{k+1} \quad .$$

Substituting from (2.13) we get

$$z_{k+1} = \alpha^{k+1} [A\ x_k \ + B\ u_k]$$

$$z_{k+1} = \alpha\ [A\ \alpha^k x_k \ + B\ \alpha^k u_k]$$

$$z_{k+1} = \alpha\ [A\ z_k + B\ v_k]$$

$$z_{k+1} = (\alpha A)\ z_k \ + (\alpha B)\ v_k$$

$$z_{k+1} = A_\alpha\ z_k \ + B_\alpha\ v_k \qquad ----(2.34)$$

where , $A_\alpha = \alpha\ A$ and $B_\alpha = \alpha\ B$ ----(2.35).

Now, for the system (2.34), with performance index (2.32) as stated in the earlier section, the optimal control law is given by

$$v_k = -\ K_\alpha\ z_k \qquad ----(2.36)$$

where

$$K_\alpha = (R + B_\alpha^T P_\alpha\ B_\alpha)^{-1} B_\alpha^T\ P_\alpha\ A_\alpha \qquad ----(2.37)$$

and, P_α is the p.d., symmetric unique real solution to the DARE

$$P_\alpha - A_\alpha^T\ P_\alpha\ A_\alpha + A_\alpha^T\ P_\alpha\ B_\alpha\ K_\alpha = Q \qquad ----(2.38).$$

Now working on (2.36) backwards, we get

$$\alpha^k\ u_k \ = -\ K_\alpha\ \alpha^k\ x_k$$

or $$u_k \ = -\ K_\alpha\ x_k \qquad ----(2.39).$$

We conclude from all this that the use of control law (2.39) in (2.13) optimizes the performance index J_α in (2.31). Furthermore, the state trajectory also satisfies (2.36) where z_k is a state vector so that x_k must decay at least as fast as $(1/\alpha^k)$ or

Thus, in other words, K_α minimizes J_α and guarantees the CL poles inside a circle of radius $1/\alpha$ centered at origin on z-plane (Note that $0 < 1/\alpha < 1$).

2.6 Conclusions

It was seen that the CLQR possesses excellent robustness properties. The robustness properties for the DLQR are not as good as those for CLQR. Nevertheless, DLQR does possess guaranteed stability margins. These, however, are dependent on the system parameters and the cost function matrices, unlike the case of CLQR. Certain modification in the performance indices gives rise to an optimal system with prescribed degree of stability.

CHAPTER # 3

ROBUSTNESS WITH PRESCRIBED DEGREE OF STABILITY

.1 Introduction

The advantages of LQR with prescribed degree of tability over that without a prescribed degree of stability are nown [5]. The extra robustness that a CLQR with prescribed degree f stability possesses, has been exploited in [6]. We will briefly eview this development. A similar development is attempted fterwards for the case of DLQR with prescribed degree of tability.

.2 Robustness Of CLQR With Prescribed Degree Of Stability

Recently in [6], a robust control design algorithm is iven. This uses the robustness properties of the CLQR with rescribed degree of stability. A sufficient condition is derived o show that a nominally optimal control law designed with adequate legree of stability remains optimal (in the sense of the definition iven in Chapter 2), for perturbed system with fixed perturbations n the state matrix. This necessarily means that the perturbed ystem is suboptimal as far as the design performance index is onsidered, but it is optimal for some other quadratic performance ndex (without prescribed degree of stability) with different values for the weighting matrices. This means that the return difference matrix of the perturbed (actual) CL system satisfies the

properties of infinite gain margin, 50 % gain reduction tolerance, and phase margin of $\pm 60^{\circ}$.

The development goes as follows:

Consider the LTI system,

$$\dot{x} = F x + G u \quad ; x(o) = x_o \qquad ----(3.1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$,

$F \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{n \times m}$ are actual constant state and constant input matrices respectively.

The corresponding nominal (design) system is described by

$$\dot{x} = F_o x + G u \qquad ----(3.2).$$

The perturbed state matrix F and the nominal state matrix F_o are related by,

$$F = F_o + \Delta F \qquad ----(3.3)$$

where ΔF is the fixed perturbation in the state matrix.

An optimal control law L_ε is designed for the nominal system (3.2), to minimize the performance index,

$$J_{c\varepsilon} = {}_{o}\int^{\infty} e^{2\varepsilon t} [\ x^T Q_c x + u^T R_c u\ ]\ dt \qquad ----(3.4)$$

where $Q_c \geq 0$; $R_c > 0$; and ε is a positive scalar.

We assume that (F,G) is controllable and (F, D_c) is observable, where $D_c^T D_c = Q$.

The optimal control law which minimizes (3.4) subject to (3.2) is given by

$$u = - L_\varepsilon x \qquad ----(3.5)$$

where

$$L_\varepsilon = R_c^{-1} G^T M_\varepsilon \qquad ----(3.6).$$

$$F_\epsilon^T M_\epsilon + M_\epsilon^T F_\epsilon - M_\epsilon G R_c^{-1} G^T M_\epsilon + Q_c = 0 \qquad ----(3.7)$$

where $F_\epsilon = F_o + \epsilon I_n$ ----(3.8)

I_n is the identity matrix of dimension n x n.

The use of optimal control law (3.5) in the nominal system (3.2) results in a CL system,

$$\dot{x} = (F_o - G L_\epsilon) x \qquad ----(3.9).$$

The CL system (3.9) has a degree of stability ϵ.

However, when used for the actual system, it results in a suboptimal performance and the corresponding cost is given by

$$J_a = x_o^T M x_o \qquad ----(3.10).$$

Assuming that the perturbed CL system,

$$\dot{x} = (F - G L_\epsilon) x \qquad ----(3.11)$$

is stable, M is obtained as a unique, p.d. symmetric real solution to the Lyapunov equation,

$$\hat{F}_\epsilon^T M + M^T \hat{F}_\epsilon + L_\epsilon^T R_c L_\epsilon + Q_c = 0 \qquad ----(3.12)$$

or equivalently of

$$\hat{F}^T M + M^T \hat{F} + L_\epsilon^T R_c L_\epsilon + Q_c + 2 \epsilon M = 0 \qquad ----(3.13)$$

where $\hat{F}_\epsilon = F + \epsilon I_n - G L_\epsilon$ ----(3.14)

and $\hat{F} = F - G L_\epsilon$ ----(3.15).

The following theorem appears as the main result in [6].

<u>Theorem 3.1</u>: The control law L_ϵ remains optimal for the actual system (for some other pair of performance index matrices Q'_c and R'_c) ; or equivalently,

$$[I + H(j\omega)]^* [I + H(j\omega)] \geq I \text{ holds for all } \omega,$$

if

least p.s.d.).

The proof of the theorem is presented in brief to get the idea of the development.

We have

$$\hat{F}^T M_\varepsilon + M_\varepsilon^T \hat{F} + L_\varepsilon^T R_C L_\varepsilon = E_\varepsilon$$

$$\rightarrow \quad (F - G L_\varepsilon)^T M_\varepsilon + M_\varepsilon^T (F - G L_\varepsilon) + L_\varepsilon^T R_C L_\varepsilon = E_\varepsilon .$$

Now treating this along the lines of equation (2.5) up to equation (2.10-a), this leads to

$$[I+ H(j\omega)]^* [I+ H(j\omega)] = I - Z(-j\omega) E_\varepsilon Z(j\omega) \qquad \text{----(3.17)}$$

where $H(j\omega) = R_C^{-1/2} G^T (j\omega I - F)^{-1} G R_C^{-1/2}$

and $Z(j\omega) = B (j\omega I - F)^{-1}$.

From this, it is observed that if E_ε is at least n.s.d., then

$$[I+ H(j\omega)]^* [I+ H(j\omega)] \geq I \qquad \text{----(3.18).}$$

E_ε can be further simplified to

$$E_\varepsilon = \Delta F^T M_\varepsilon + M_\varepsilon \Delta F - (Q_C + 2 \varepsilon M_\varepsilon) \qquad \text{----(3.19).}$$

Thus a condition on ΔF and ε is found out (viz. E_ε of (3.19) should be at least n.s.d.). When this condition is satisfied, the actual system (3.1) with control law of (3.6) is optimal for some other pair of performance index matrices, Q_C' and R_C'. Significantly, whatever are Q_C' and R_C', the system inherently possesses the important robustness properties of CLQR which are derived from the return difference inequality of (3.18).

In this way, a spectral factorisation of the RDM of the actual system is achieved. This needs the above-mentioned condition on E_ε to be be satisfied.

the perturbation matrix ΔF.

The main aim of our thesis has been - to find out a similar condition in case of discrete-time systems.

3.3 Robustness Of DLQR With Prescribed Degree Of Stability

A discrete-time analogy of the above result [of section (3.2)] was tried. In continuous time case, the spectral factorisation of the RDM of the actual system was found out. In that, the perturbations in the state matrix F_o were considered. Apparently it is observed that such a result is possible for the continuous-time case, because the expression for the control law viz., $L_\varepsilon = R_c^{-1} G^T M_\varepsilon$ does not explicitly involve the matrix F_o whose perturbations are considered.

On the other hand, in the discrete-time case, the expression for optimal control law is

$$K_\alpha = [\ R + B_\alpha^T\ P_\alpha\ B_\alpha]^{-1} B_\alpha^T\ P_\alpha\ A_\alpha$$

which explicitly involves A (since, $A_\alpha = \alpha A$). Hence in the discrete-time case, deriving the spectral factorisation of the RDM for the actual system [e.g. equation (3.17) for CLQR], from a Riccati-type equation [equation (3.16) in case of CLQR] was not possible.

One more important difference between the CLQR and the DLQR is the following. In case of the CLQR, the return difference inequality gives rise to the stability margins for the CL system, which are independent of the system parameters, viz. for any LQR, (with whatever values of F, G, Q_c and R_c) the gain margins are 1/2, ∞

gain margins for the system described by (2.13)-(2.18), are given by

downward gain margin = $(1 + \Lambda)^{-1}$

upward gain margin = $(1 - \Lambda)^{-1}$

and the phase margins are $\pm 2 \sin^{-1}(\Lambda / 2)$.

Here, $\Lambda = \{ \det[I + R^{-1/2} B^T P B R^{-1/2}] \}^{-1/2}$, which is clearly dependant on all system parameters; which makes the problem more complex.

Hence, first the stability margins for the DLQR with prescribed degree of stability α, ($\alpha > 1$) are developed and these are compared with those without any prescribed degree of stability ($\alpha = 1$).

Consider the system of (2.3) viz.

$$x_{k+1} = A x_k + B u_k \qquad ----(3.20).$$

We know that for minimizing,

$$J_\alpha = \sum_{i=1}^{\infty} \alpha^{2k} [x_k^T Q_k x + u_k^T R u_k] \qquad ----(3.21)$$

the control law obtained is

$$u_k = - K_\alpha x_k \qquad ----(3.22).$$

$$K_\alpha = (R + B_\alpha^T P_\alpha B_\alpha)^{-1} B_\alpha^T P_\alpha B_\alpha \qquad ----(3.23)$$

where

$$A_\alpha = \alpha A \quad \text{and,} \quad B_\alpha = \alpha B \qquad ----(3.24).$$

Hence

$$K_\alpha = \alpha^2 (R + B^T \alpha^2 P_\alpha B)^{-1} B^T P_\alpha A \qquad ----(3.25).$$

P_α is the unique, p.d., symmetric, real solution to the DARE,

$$\rightarrow \qquad \frac{P_\alpha}{\alpha^2} - A^T P_\alpha A + A^T P_\alpha B K_\alpha = \frac{Q}{\alpha^2} .$$

Adding $P_\alpha - P_\alpha / \alpha^2$ to both sides gives

$$P_\alpha - A^T P_\alpha A + A^T P_\alpha B K_\alpha = Q / \alpha^2 + P_\alpha (1 - 1 / \alpha^2) .$$

Define $Q_1 \triangleq Q / \alpha^2 + P_\alpha (1 - 1 / \alpha^2)$

$$\rightarrow \qquad P_\alpha - A^T P_\alpha A + A^T P_\alpha B K_\alpha = Q_1 .$$

Add `$z(A^T P - A^T P) + z^{-1}(P A - P A)$' to the L.H.S. to get

$$z(A^T P_\alpha - A^T P_\alpha) + z^{-1}(P_\alpha A - P_\alpha A) + P_\alpha - A^T P_\alpha A + A^T P_\alpha B K_\alpha = Q_1 .$$

Rearranging , we get

$$(z^{-1} I - A^T P_\alpha) P_\alpha (zI - A) + A^T P_\alpha (zI - A)$$
$$+ (z^{-1} I - A^T P_\alpha) P_\alpha A + A^T P_\alpha B K_\alpha = Q_1 .$$

Now multiply on the left by $\Phi^* = (z^{-1} I - A^T)^{-1}$ and on the right by $\Phi = (zI - A)^{-1}$, to obtain

$$P_\alpha + \Phi^* A^T P_\alpha + P_\alpha A \Phi + \Phi^* A^T P_\alpha B K_\alpha \Phi = \Phi^* Q_1 \Phi .$$

Also multiplying on the left by B_α^T and on the right by B_α gives

$$B_\alpha^T P_\alpha B_\alpha + B_\alpha^T \Phi^* A^T P_\alpha B_\alpha + B_\alpha^T P_\alpha A \Phi B_\alpha + B_\alpha^T \Phi^* A^T P_\alpha B K_\alpha \Phi B_\alpha = B_\alpha^T \Phi^* Q_1 \Phi B_\alpha .$$

Using (3.24), we have

$$B_\alpha^T P_\alpha B_\alpha + \alpha B^T \Phi^* A^T P_\alpha B_\alpha + B_\alpha^T P_\alpha A \Phi \alpha B + \alpha B^T \Phi^* A^T P_\alpha B K_\alpha \Phi \alpha B$$
$$= B_\alpha^T \Phi^* Q_1 \Phi B_\alpha .$$

$$\rightarrow \quad B_\alpha^T P_\alpha B_\alpha + B^T \Phi^* A_\alpha^T P_\alpha B_\alpha + B_\alpha^T P_\alpha A_\alpha \Phi B + B^T \Phi^* A_\alpha^T P_\alpha B_\alpha K_\alpha \Phi B$$
$$= B_\alpha^T \Phi^* Q_1 \Phi B_\alpha .$$

Add R to both sides to obtain

$$R + B_\alpha^T P_\alpha B_\alpha + B^T \Phi^* A_\alpha^T P_\alpha B_\alpha + B_\alpha^T P_\alpha A_\alpha \Phi B + B^T \Phi^* A_\alpha^T P_\alpha B_\alpha K_\alpha \Phi B$$
$$= R + B_\alpha^T \Phi^* Q_1 \Phi B_\alpha .$$

$$+(R + B_\alpha^T P_\alpha B_\alpha)(R + B_\alpha^T P_\alpha B_\alpha)^{-1} B_\alpha^T P_\alpha A_\alpha \Phi\ B$$
$$+ B^T \Phi^* A_\alpha^T P_\alpha B_\alpha (R + B_\alpha^T P_\alpha B_\alpha)^{-1}(R + B_\alpha^T P_\alpha B_\alpha) K \Phi\ B$$
$$= R + W^* Q_1 W$$

where $W = \Phi\ B_\alpha$.

This simplifies to

$$(I + B^T \Phi^* K_\alpha^T)(R + B_\alpha^T P_\alpha B_\alpha)(I + K_\alpha \Phi\ B) = R + W^* Q_1 W \qquad \text{----(3.27)}.$$

(3.27) is the return difference equality for the system (3.20)-(3.21). Henceforth we would use a notation to refer to this system as (A,B,K_α).

The return difference of this system is

$$F(z) = (I + K_\alpha \Phi\ B) \qquad \text{----(3.28)}.$$

From equation (3.27), we find out that for this system a necessary condition for optimality derived on the similar lines of the reference [16] is given by,

$$|\det [F(z)]| \geq \{ \det[\ I + R^{-1/2} B_\alpha^T P_\alpha\ B_\alpha\ R^{-1/2}\ ] \}^{-1/2}$$
$$= \{ \det[I + R^{-1/2} B^T \alpha^2 P_\alpha B\ R^{-1/2}] \}^{-1/2} \triangleq \Lambda_\alpha \qquad \text{--(3.29)}.$$

It was not possible to establish an explicit and useful relationship between P_α of (3.29) and P of (2.17), involved in Λ for the system (A,B,K) viz.

$$\Lambda \triangleq \{ \det[\ I + R^{-1/2} B^T P\ \ B\ \ R^{-1/2}\ ]\ \}^{-1/2} \qquad \text{--(3.30)}.$$

But it is verified from some numerical examples (some of them to follow) that

$$\alpha^2 P_\alpha > P \qquad \text{----(3.31)}$$

so that,

$$\Lambda_\alpha < \Lambda \qquad \text{----(3.32)}.$$

In [18], it is shown that for the system (A,B,K) if we

then the gain margins of the system are

$$\text{downward gain margin} = [\ (1 + \varphi) - (1 + \varphi)^{1/2}\ ] / \varphi \quad \text{---(3.34)}$$

$$\text{upward gain margin} = [\ (1 + \varphi) + (1 + \varphi)^{1/2}\ ] / \varphi \quad \text{--(3.35)}.$$

The largest margins correspond to the lowest value of φ. Now, since we have

$$\alpha^2 P_\alpha > P$$

hence

$$B^T \alpha^2 P_\alpha B \ > \ B^T P \ B$$

which again means that the margins obtained in case of $\alpha > 1$ are smaller than those for $\alpha = 1$.

A similar result comes out if the margins are derived as in [19]. It is, however, observed that the margins derived above for $\alpha > 1$ are more conservative than those for $\alpha = 1$. This is evident from the following facts:

We have for $\alpha = 1$

$$(I + B^T \Phi^* K^T)(R + B^T P\ B)(I + K\ \Phi\ B) = R + W^* Q\ W \quad \text{----(3.36)}.$$

Since Q is p.s.d., for $|z| = 1$, $W^* Q\ W$ is also p.s.d. and hence we get,

$$(I + B^T \Phi^* K^T)(R + B^T P\ B)(I + K\ \Phi\ B) \geq R \quad \text{----(3.37)}.$$

The margins are calculated using the inequality of (3.37). For the case of $\alpha > 1$, we have

$$(I + B^T \Phi^* K_\alpha^T)(R + B_\alpha^T P_\alpha B_\alpha)(I + K_\alpha \Phi\ B) = R + B_\alpha^T \Phi^* Q_1 \Phi\ B_\alpha \ .$$

Replacing Q_1 and using (3.24) we get

$$(I + B^T \Phi^* K_\alpha^T)(R + B^T \alpha^2 P_\alpha B\ )(I + K_\alpha \Phi\ B)$$

$$= R + B^T \Phi^* [\ Q + P_\alpha(\alpha^2 - 1)]\ \Phi\ B \quad \text{----(3.38)}.$$

Now, the term $[\ Q + P_\alpha(\alpha^2-1)]$ is <u>positive definite</u> and

$$(I + B^T\Phi^* K_\alpha^T)(R + B^T\alpha^2 P_\alpha B)(I + K_\alpha \Phi B) \geq R \qquad ----(3.40)$$

and hence the conservativeness of the margins increases.

In what follows, we analyze the results in a different perspective. Here, it is proved that the margins derived for $\alpha > 1$ are the margins before the CL poles of the system come out of the circle of radius $1/\alpha$, centered at the origin on the z-plane. In that sense, these margins are the measure of the nearness of the CL poles to the circle of radius $1/\alpha$ and hence the stability margins derived for $\alpha > 1$ are with respect to the circle of radius $1/\alpha$ centered at the origin.

We know that for the system of (2.13), with the performance index J of (2.14) the return difference equality is

$$(I + K \Phi B)^*(R + B^T P B)(I + K \Phi B) = R + W^* Q W \qquad ----(3.41).$$

Now consider a system,

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \qquad ----(3.42)$$

where

$$A_\alpha = \alpha A \quad \text{and,} \quad B_\alpha = \alpha B \quad .$$

For minimizing,

$$J = \sum_{i=1}^{\infty} [x_k^T Q x_k + u_k^T R u_k] \qquad ----(3.43)$$

the optimal controller is given by

$$K_\alpha = (R + B_\alpha^T P_\alpha B_\alpha)^{-1} B_\alpha^T P_\alpha A_\alpha \qquad ----(3.44).$$

P_α is the unique, p.d., symmetric ,real solution to the DARE,

$$P_\alpha - A_\alpha^T P_\alpha A_\alpha + A_\alpha^T P_\alpha B_\alpha K_\alpha = Q \qquad ----(3.45).$$

$$(I + K_\alpha \Phi_\alpha B_\alpha)^*(R + B_\alpha^T P_\alpha B_\alpha)(I + K_\alpha \Phi_\alpha B_\alpha) = R + B_\alpha^T \Phi_\alpha^* Q\, \Phi_\alpha B_\alpha \quad \text{---(3.46)}$$

where

$$\Phi_\alpha = (zI - A_\alpha)^{-1} = (zI - \alpha A)^{-1} \quad \text{----(3.47).}$$

This means,

$$[I + K_\alpha(zI - \alpha A)^{-1}\alpha B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(zI - \alpha A)^{-1}\alpha B]$$

$$= R + \alpha^2 B^T(z^{-1}I - \alpha A)^{-T} Q (zI - \alpha A)^{-1}B \qquad \text{on } |z| = 1$$

$$\rightarrow \quad [I + K_\alpha(\tfrac{z}{\alpha}I - A)^{-1}\alpha^{-1}\alpha B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(\tfrac{z}{\alpha}I - A)^{-1}\alpha^{-1}\alpha B]$$

$$= R + \alpha^2[(\tfrac{z}{\alpha}I - A)^{-1}\alpha^{-1}B]^* Q (\tfrac{z}{\alpha}I - A)^{-1}\alpha^{-1}B \qquad \text{on } |z| = 1$$

$$\rightarrow \quad [I + K_\alpha(\tfrac{z}{\alpha}I - A)^{-1}B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(\tfrac{z}{\alpha}I - A)^{-1}B]$$

$$= R + [(\tfrac{z}{\alpha}I - A)^{-1}B]^* Q (\tfrac{z}{\alpha}I - A)^{-1}B \qquad \text{on } |z| = 1$$

which is equivalent to saying that

$$[I + K_\alpha(zI - A)^{-1}B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(zI - A)^{-1}B]$$

$$= R + [(zI - A)^{-1}B]^* Q [(zI - A)^{-1}B] \qquad \underline{\text{on } |z| = 1/\alpha}$$

i.e.

$$(I + K_\alpha \Phi B)^*(R + B^T\alpha^2 P_\alpha B)(I + K_\alpha \Phi B)$$

$$= R + B^T\Phi^* Q\, \Phi B \qquad \underline{\text{on } |z| = 1/\alpha} \quad \text{----(3.48)}$$

where, $\Phi = (z\ I - A\ )^{-1}$.

Note that $(I + K_\alpha \Phi B)$ is the return difference for the system (A,B,K_α).

Hence we have for (A,B,K_α),

$$|\det[I + K_\alpha \Phi B]|$$

$$\geq \{\det[I + R^{-1/2}B^T\alpha^2 P_\alpha B\ R^{-1/2}]\}^{-1/2} = \Lambda_\alpha \quad \text{----(3.49)}$$

which is also valid on $|z| = 1/\alpha$.

This means that the gain and phase margins derived from Λ_α are valid on the circle $|z| = 1/\alpha$, which is to say that the system has these margins, before its CL poles come out of the

Closed loop poles are
(-.15049,-.39307)
(-.15049,.393072)
(-8.67361E-19,-6.93889E-18)

Alpha = 1.0

DET. OF RDM ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.985428	.000000	.985428
2	12.9	.994358	.049554	.995592
3	25.7	1.023516	.099875	1.028377
4	38.6	1.081302	.150028	1.091661
5	51.4	1.186778	.191820	1.202180
6	64.3	1.376304	.186463	1.388878
7	77.1	1.658724	-.019033	1.658833
8	90.0	1.645691	-.601975	1.752334
9	102.9	1.083050	-.821697	1.359479
10	115.7	.750966	-.619305	.973391
11	128.6	.638847	-.420848	.765009
12	141.4	.601877	-.277471	.662756
13	154.3	.589278	-.169581	.613193
14	167.1	.585090	-.080692	.590628
15	180.0	.584111	.000000	.584111

Min value of Det. = .5841112732887268

****
Alpha = 1.2
****
Lower Bound on DET(I + K * PHI * B)
[Equation (3.29)]
0.2372425159798326

Solution to the DARE is

$$\begin{bmatrix} 1.26434982596346 & -.20060347871451 & -.528699651926912 \\ -.20060347871451 & 2.09150358738945 & .4012069574290193 \\ -.528699651926912 & .401206957429019 & 1.05739930385382 \end{bmatrix}$$

Eigen-values of alp^2 Palpha - P are
(.294124,.0)
(.738923,.0)
(1.5653,.0)

The LQR controller is

$$\begin{bmatrix} 5.170192750373961E-02 & .4609526836625133 & -.103403855007479 \\ -.276394245708663 & -.229343946297915 & .5527884914173254 \end{bmatrix}$$

Closed loop poles are
(-.13587,.301214)

Alpha = 1.2

DET. OF RDM ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.920622	.000000	.920622
2	12.9	.930210	.064776	.932462
3	25.7	.961664	.131623	.970630
4	38.6	1.024664	.201217	1.044234
5	51.4	1.141805	.267433	1.172706
6	64.3	1.359321	.292303	1.390393
7	77.1	1.708833	.106297	1.712136
8	90.0	1.781618	-.543485	1.862670
9	102.9	1.189094	-.865340	1.470631
10	115.7	.794742	-.683090	1.047963
11	128.6	.647113	-.474208	.802265
12	141.4	.591820	-.316277	.671031
13	154.3	.569617	-.194556	.601927
14	167.1	.560707	-.092888	.568349
15	180.0	.558299	.000000	.558299

Min value of Det. = .5582988858222961

DET. OF RDM ON |Z| = 1.0/alp

Sno.	Th	Det		|Det|
1	.0	.862399	.000000	.862399
2	12.9	.870229	.079501	.873852
3	25.7	.896228	.165701	.911417
4	38.6	.950019	.267413	.986938
5	51.4	1.058213	.398399	1.130724
6	64.3	1.304625	.576440	1.426299
7	77.1	2.025542	.702058	2.143760
8	90.0	3.009875	-1.164611	3.227332
9	102.9	.896093	-1.507618	1.753823
10	115.7	.520335	-.841560	.989430
11	128.6	.474390	-.517576	.702091
12	141.4	.473073	-.329331	.576417
13	154.3	.477945	-.198493	.517524
14	167.1	.481838	-.093932	.490908
15	180.0	.483223	.000000	.483223

Min value of Det. = .4832231700420379
Bound on det[I+K*PHI*B] : .2372425159798326

****

Alpha = 2.0

****

Lower Bound on DET(I + K * PHI * B)
[Equation (3.29)]
6.499572522470705E-02

Solution to the DARE is

$$\begin{bmatrix} 1.66232671568697 & -.928022924168846 & -1.32465343137393 \\ -.928022924168846 & 3.77876957085853 & 1.85604584833769 \\ -1.32465343137393 & 1.85604584833769 & 2.64930686274786 \end{bmatrix}$$

Eigen-values of alp^2 Palpha - P are
(21.2166,.0)
(2.1094,.0)
(5.2788,.0)

The LQR controller is

$$\begin{bmatrix} .103113658240982\,9 & .580136714349052\,4 & -.206227316481966 \\ -.293516848964102 & -.342587008171782 & .587033697928203\,4 \end{bmatrix}$$

Closed loop poles are
(-7.06665E-02,-.11839)
(-7.06665E-02,.11839)
(1.58203E-16,-2.75143E-18)

Alpha = 2.0

DET. OF RDM ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.773562	.000000	.773562
2	12.9	.782811	.089713	.787934
3	25.7	.813513	.184358	.834140
4	38.6	.876554	.288533	.922820
5	51.4	.998749	.402618	1.076848
6	64.3	1.241625	.498611	1.338001
7	77.1	1.687722	.400535	1.734599
8	90.0	1.961980	-.282666	1.982238
9	102.9	1.417406	-.795312	1.625288
10	115.7	.947988	-.701858	1.179528
11	128.6	.741460	-.510161	.900015
12	141.4	.651432	-.348376	.738735
13	154.3	.609610	-.217089	.647110
14	167.1	.590633	-.104334	.599777
15	180.0	.585118	.000000	.585118

Min value of Det. = .5851179957389831

DET. OF RDM ON |Z| = 1.0/alp

Sno.	Th	Det		|Det|
1	.0	.452902	.000000	.452902
2	12.9	.447387	.104334	.459392
3	25.7	.428410	.217089	.480274
4	38.6	.386588	.348376	.520400
5	51.4	.296560	.510161	.590095
6	64.3	.090032	.701859	.707609
7	77.1	-.379386	.795311	.881166
8	90.0	-.923960	.282666	.966231
9	102.9	-.649701	-.400534	.763243
10	115.7	-.203605	-.498611	.538579
11	128.6	.039271	-.402618	.404529
12	141.4	.161466	-.288533	.330640
13	154.3	.224507	-.184358	.290502
14	167.1	.255209	-.089713	.270518
15	180.0	.264458	.000000	.264458

Min value of Det. = .2644580006599426
Bound on det[I+K*PHI*B] : 6.499572522470705E-02

****
Alpha = 2.8
****

Lower Bound on DET(I + K * PHI * B)
[Equation (3.29)]
2.502080204346506E-02

Solution to the DARE is

$$\begin{bmatrix} 2.26554678694545 & -2.11180601036565 & -2.5310935738909 \\ -2.11180601036565 & 6.2302038385915 & 4.2236120207313 \\ -2.5310935738909 & 4.2236120207313 & 5.06218714778179 \end{bmatrix}$$

Eigen-values of alp^2 Palpha - P are

(85.4056,.0)
(4.8962,.0)
(12.2357,.0)

The LQR controller is

$$\begin{bmatrix} .1266070749619696 & .6264865804201737 & -.253214149923939 \\ -.292680488563275 & -.380105237440033 & .585360977126550 3 \end{bmatrix}$$

Closed loop poles are
(-3.93243E-02,6.06199E-02)
(2.32019E-17,1.73472E-17)
(-3.93243E-02,-6.06199E-02)

Alpha = 2.8

DET. OF RDM ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.722580	.000000	.722580
2	12.9	.731212	.095736	.737453
3	25.7	.760026	.197370	.785235
4	38.6	.819865	.310933	.876845
5	51.4	.937969	.439550	1.035852
6	64.3	1.179226	.560891	1.305822
7	77.1	1.643617	.505222	1.719513
8	90.0	1.989558	-.157297	1.995766
9	102.9	1.493192	-.724907	1.659853
10	115.7	1.015823	-.673993	1.219083
11	128.6	.794793	-.499396	.938665
12	141.4	.694590	-.344234	.775211
13	154.3	.646558	-.215585	.681553
14	167.1	.624269	-.103873	.632852
15	180.0	.617715	.000000	.617715

Min value of Det. = .6177150011062622

DET. OF RDM ON |Z| = 1.0/alp

Sno.	Th	Det		|Det|
1	.0	.256331	.000000	.256331
2	12.9	.246924	.077941	.258933
3	25.7	.216717	.155815	.266917
4	38.6	.159513	.230950	.280682
5	51.4	.065286	.293024	.300209
6	64.3	-.073098	.314693	.323072
7	77.1	-.232954	.248309	.340477
8	90.0	-.328447	.075416	.336995
9	102.9	-.284786	-.113531	.306582
10	115.7	-.155010	-.213745	.264036
11	128.6	-.027114	-.224258	.225892
12	141.4	.065969	-.186676	.197990
13	154.3	.124862	-.129562	.179936
14	167.1	.156758	-.065727	.169980
15	180.0	.166813	.000000	.166813

Min value of Det. = .1668127179145812
Bound on det[I+K*PHI*B] : 2.502080204346506E-02

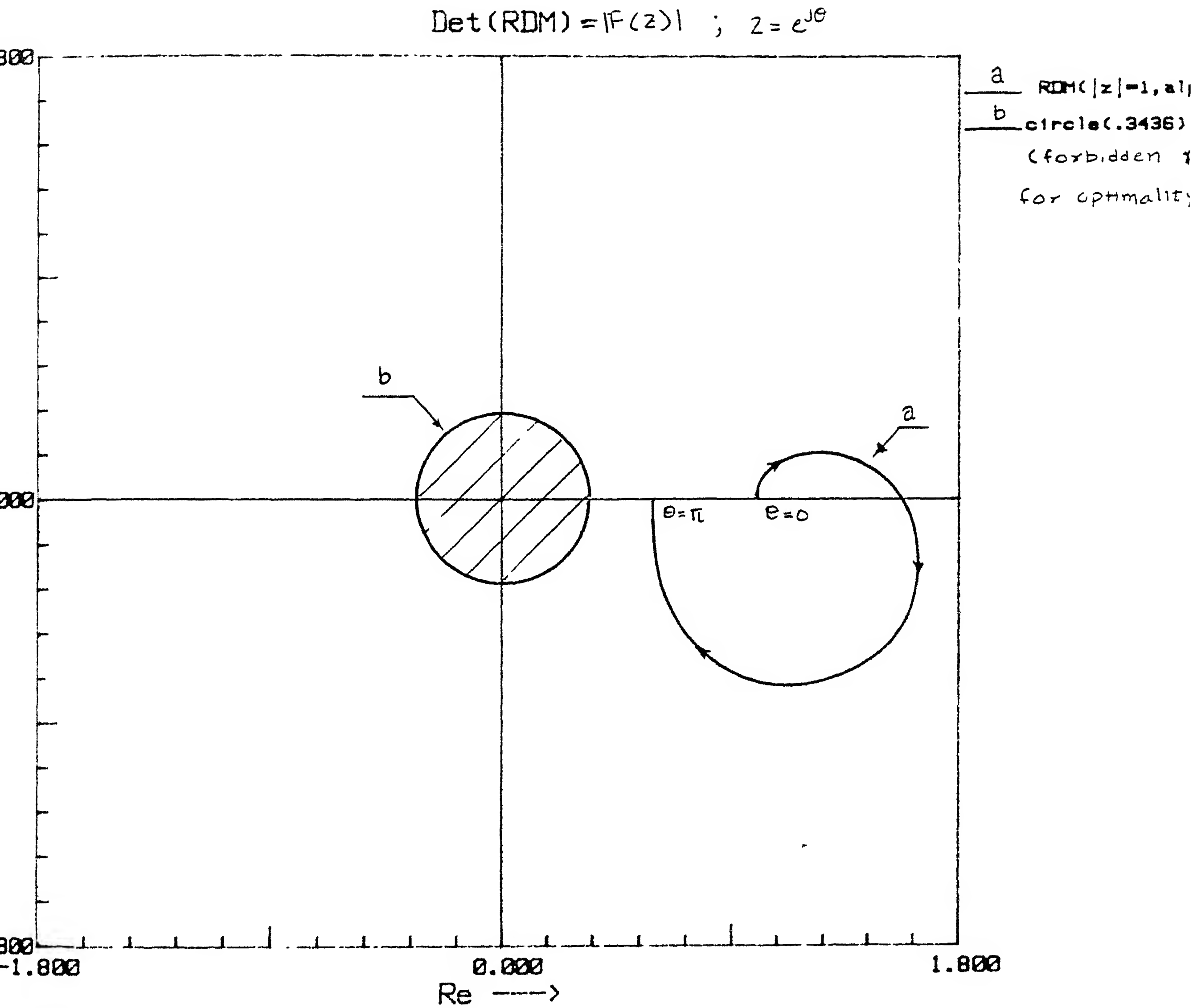


Figure 3-a :Det. of RDM on |z| = 1. (α = 1.0)

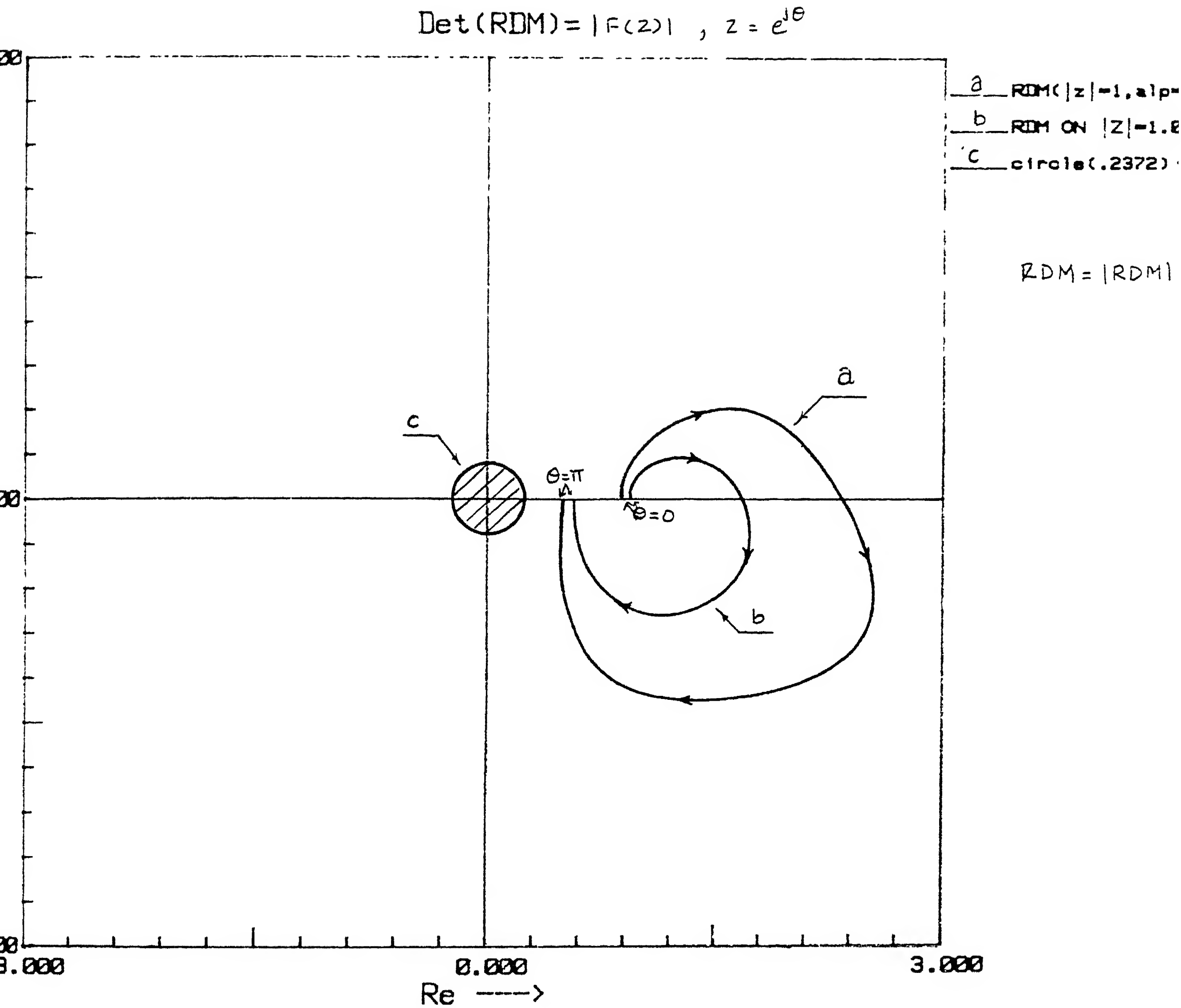


Figure 3-b : Det. of RDM for α = 1.2.

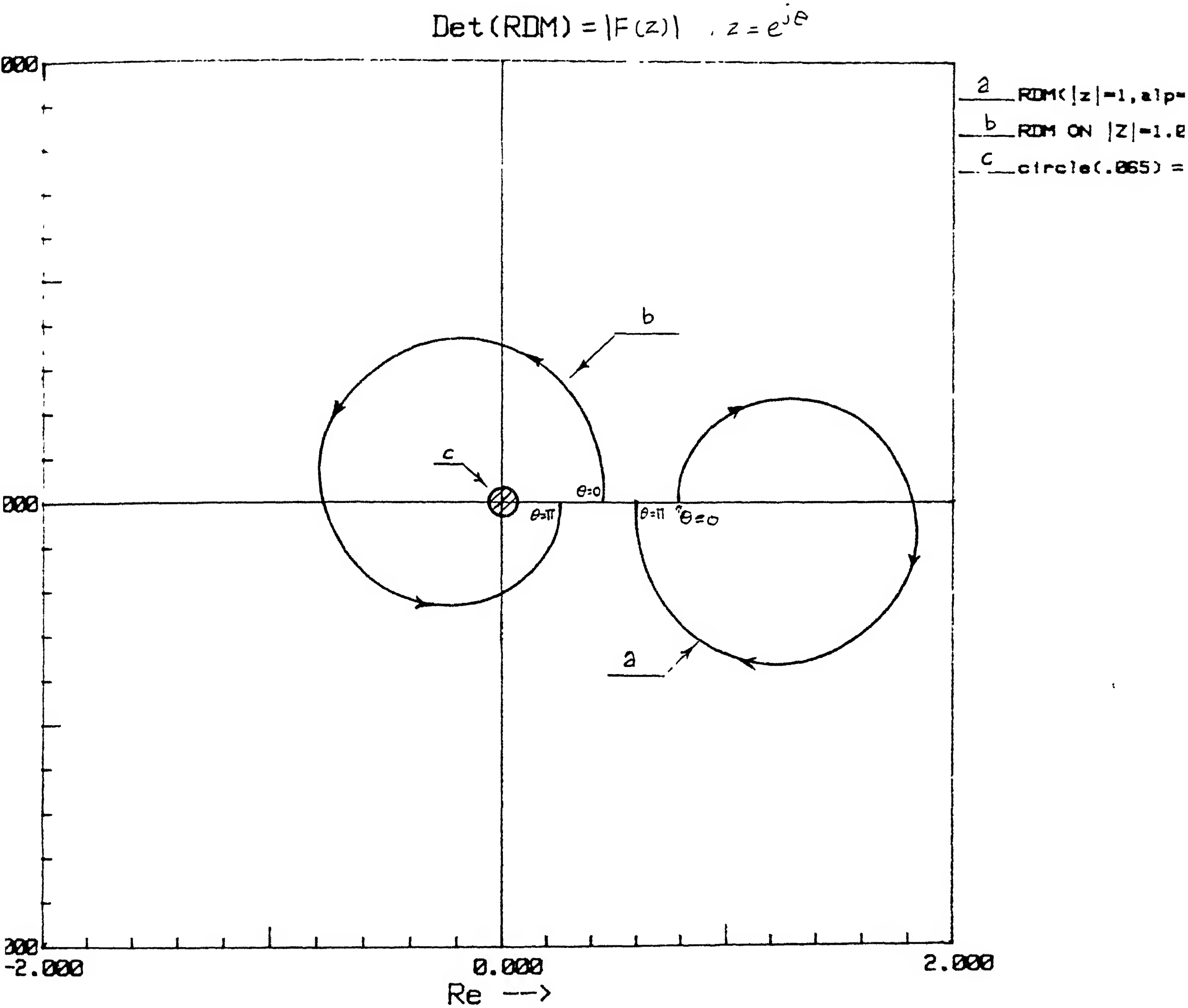


Figure 3-c : Det. of RDM for α = 2.0.

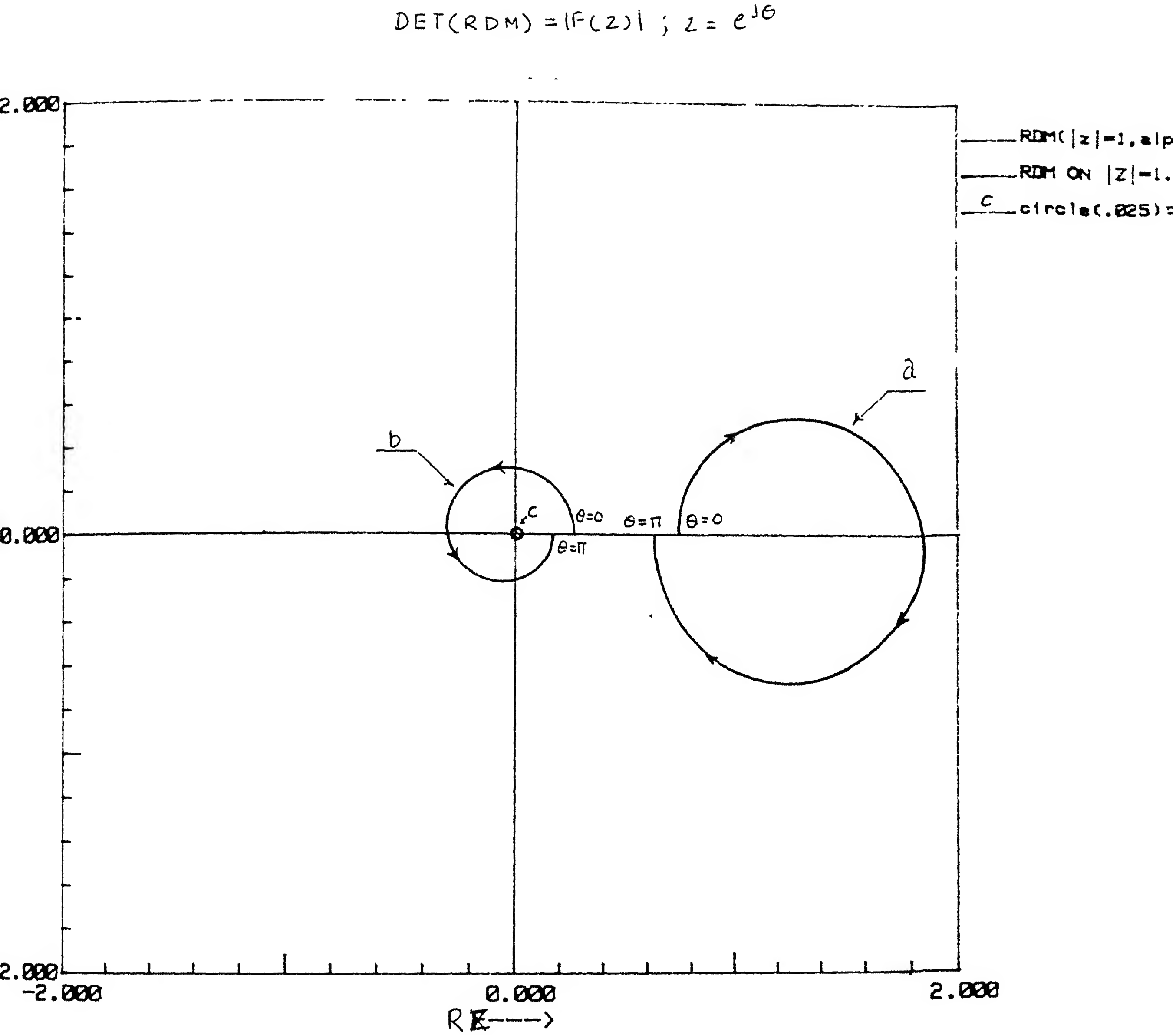


Figure 3-d : Det. of RDM for α = 2.8.

In the numerical example presented, four values of α, viz., α = 1, 1.2, 2.0 and 2.8, are considered. The bound on the minimum value of the determinant of the RDM of the system is calculated [equation (3.29)]. It is seen that this bound is satisfied by the values of the determinants of the RDMs on $|z|=1$. It is also verified that the values of the determinants of the RDMs on $|z| = 1/\alpha$, satisfy these bounds. It is also observed that the eigenvalues of $\alpha^2 P_\alpha - P$ are positive in each case so that $\alpha^2 P_\alpha > P$. The graphical representation of each case is shown in figures (3-a) - (3-d).

We have seen that, if a controller $K_{\alpha 1}$ is used for a system, the system possesses gain margins corresponding to the radius $\Lambda_{\alpha 1}$. The CL systems with lesser degree of stability $\alpha < \alpha_1$, continue to possess these stability margins, since, as seen earlier, the radius Λ_α decreases as α increases . Due to perturbations, if the CL poles of the system are pushed nearer to the unit circle, the degree of stability of the system will reduce. But still, the system will continue to possess the margins corresponding to the radius Λ_α.

Consider 3 values of α , viz., $\alpha_1 < \alpha_2 < \alpha_3$.
Then the corresponding inequalities are

$$|\det [F_1(z)]| \geq \Lambda_{\alpha 1}$$

$$|\det [F_2(z)]| \geq \Lambda_{\alpha 2}$$

$$|\det [F_3(z)]| \geq \Lambda_{\alpha 3}$$

where $F_i(z)$ indicates the RDM corresponding to the controller $K_{\alpha i}$. Now as seen earlier,

$$|\det [F_2(z)]| \geq \Lambda_{\alpha s}$$

and

$$|\det [F_1(z)]| \geq \Lambda_{\alpha 2} \geq \Lambda_{\alpha s} .$$

Thus, the systems with lesser degree of stability satisfy the necessary condition for optimality for higher degree of stability. And hence they possess guaranteed stability margins corresponding to the radius $\Lambda_{\alpha s}$.

If we replace A and B in the system of (3.20) by βA and βB respectively, where β is a positive scalar such that $1 \leq \beta \leq \alpha$, the resulting system will be

$$x_{k+1} = \beta A x_k + \beta B u_k$$

$$\triangleq A_\beta x_k + B_\beta u_k \qquad ----(3.50).$$

If for this system, a controller K_α designed for the system of (3.20) is used, then the guaranteed degree of stability will be less than α. It will be shown that this CL system continues to possess the margins corresponding to the radius Λ_α (with respect to the unit circle).

Proof: For the system of (3.50) we use

$$K_\alpha = (R + B_\alpha^T P_\alpha B_\alpha)^{-1} B_\alpha^T P_\alpha A_\alpha$$

where P_α satisfies

$$P_\alpha - A_\alpha^T P_\alpha A_\alpha + A_\alpha^T P_\alpha B_\alpha K_\alpha = Q \quad .$$

From this, we get

$$[I + K_\alpha(zI - \alpha A)^{-1}\alpha B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(zI - \alpha A)^{-1}\alpha B]$$

$$= R + \alpha^2 B^T(z^{-1}I - \alpha A)^{-T} Q (zI - \alpha A)^{-1} B \qquad \text{on } |z| = 1$$

$$[I + K_\alpha(z\frac{\beta}{\alpha}I - \beta A)^{-1}\frac{\beta}{\alpha}\alpha B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(z\frac{\beta}{\alpha}I - \beta A)^{-1}\frac{\beta}{\alpha}\alpha B]$$

$$\rightarrow [I + K_\alpha(z \tfrac{\beta}{\alpha}I - \beta A)^{-1}\beta B]^*(R + B^T\alpha^2 P_\alpha B)[I + K_\alpha(z \tfrac{\beta}{\alpha}I - \beta A)^{-1}\beta B]$$

$$= R + [(z \tfrac{\beta}{\alpha}I - \beta A)^{-1}\beta B]^* Q (z \tfrac{\beta}{\alpha}I - \beta A)^{-1}\beta B \qquad \text{on } |z| = 1$$

$$\rightarrow [I+K_\alpha(z \tfrac{\beta}{\alpha}I - A_\beta)^{-1} B_\beta]^*(R + B^T\alpha^2 P_\alpha B)[I+K_\alpha(z \tfrac{\beta}{\alpha}I - A_\beta)^{-1} B_\beta]$$

$$= R + [(z \tfrac{\beta}{\alpha}I - A_\beta)^{-1}B_\beta]^* Q (z \tfrac{\beta}{\alpha}I - A_\beta)^{-1} B_\beta \qquad \text{on } |z| = 1$$

which is equivalent to saying that

$$[I + K_\alpha(zI - A_\beta)^{-1}B_\beta]^*(R + B^T\alpha^2P_\alpha B)[I + K_\alpha(zI - A_\beta)^{-1}B_\beta]$$

$$= R + [(zI - A_\beta)^{-1}B_\beta]^* Q [(zI - A_\beta)^{-1}B_\beta] \qquad \underline{\text{on}}\ \underline{|z| = \beta/\alpha}$$

i.e.

$$(I + K_\alpha\Phi_\beta B_\beta)^*(R + B^T\alpha^2P_\alpha B)(I + K_\alpha\Phi_\beta B_\beta)$$

$$= R + B_\beta^T\Phi_\beta^* Q \Phi_\beta B_\beta \qquad \underline{\text{on}}\ \underline{|z| = \beta/\alpha}$$

where, $\Phi_\beta = (z I - A_\beta)^{-1}$.

Note that $(I + K_\alpha\Phi_\beta B_\beta)$ is the return difference for the system $(A_\beta, B_\beta, K_\alpha)$.

Hence we have for $(A_\beta, B_\beta, K_\alpha)$,

$$|\det [I + K_\alpha\Phi_\beta B_\beta]|$$

$$\geq \{ \det[I + R^{-1/2}B^T\alpha^2P_\alpha B R^{-1/2}] \}^{-1/2} = \Lambda_\alpha$$

which is valid on $|z| = \beta/\alpha$.

Note that for all values of β in the range $[1,\alpha]$, the circles $|z|=\beta/\alpha$ lie between the circles $|z|= 1$ and $|z|=1/\alpha$ (fig.3-e). Thus the CL system $(A_\beta, B_\beta, K_\alpha)$ has stability margins corresponding to the radius Λ_α and theses margins are, in fact, valid with respect to the circle $|z|= \beta/\alpha$. In the extreme case, when $\beta = \alpha$, these margins are with respect to the unit circle. We consider the same numerical example for illustrating

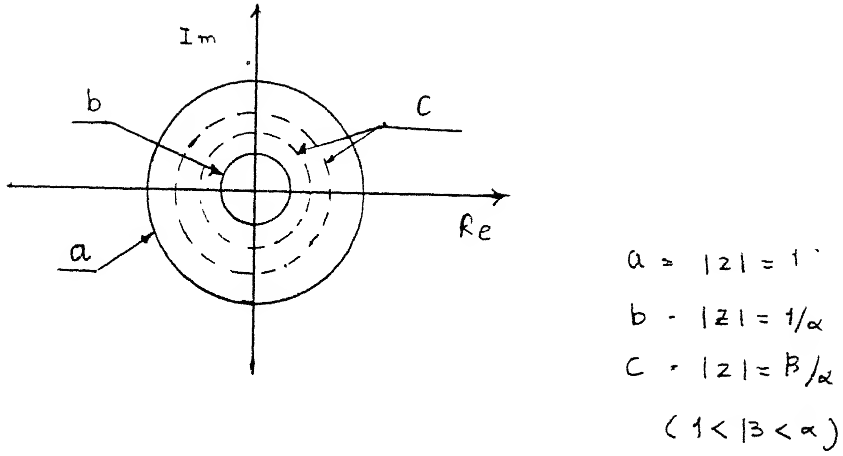
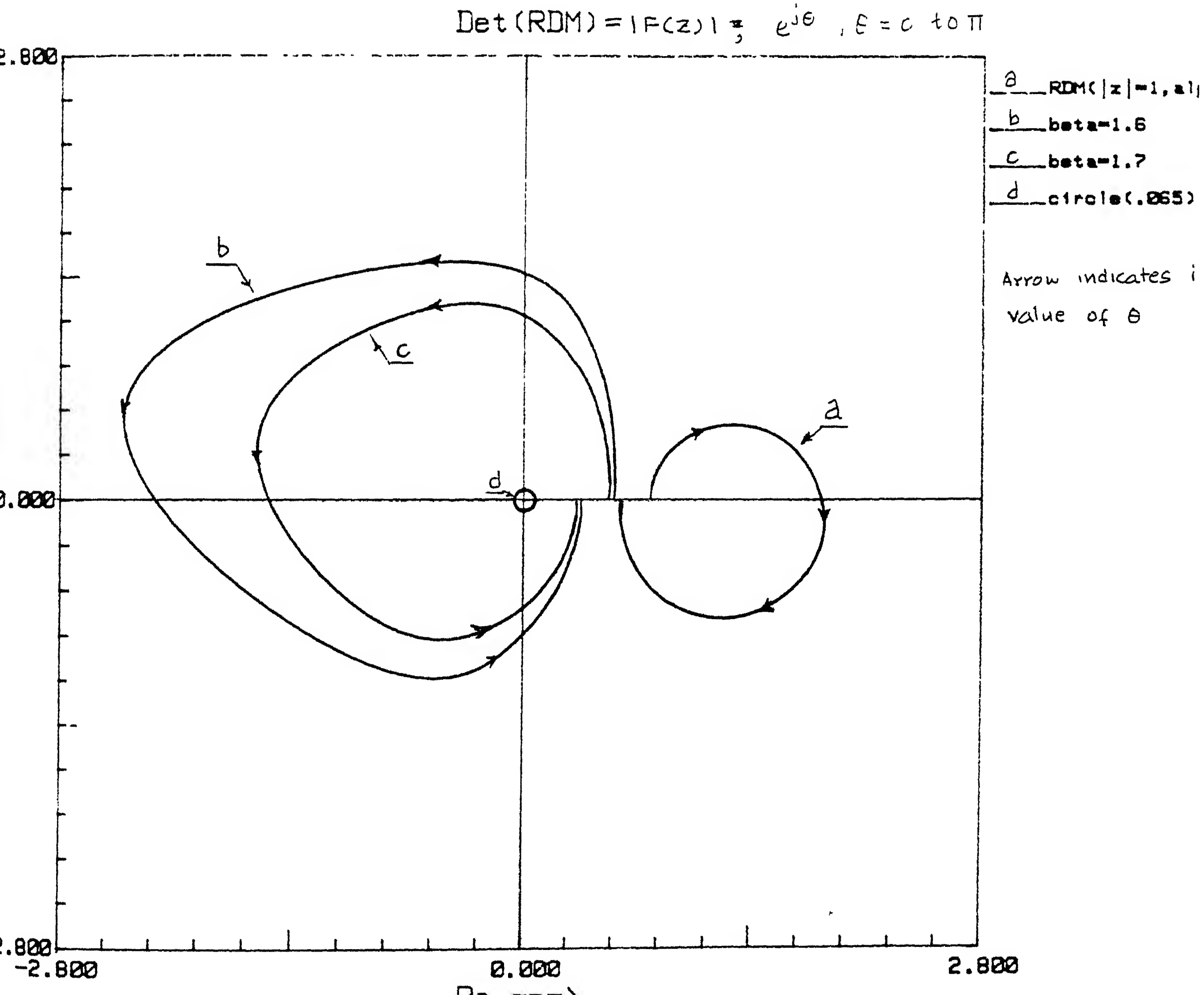


figure 3·c Circles of $|z| = \beta/\alpha$ $(1 < \beta < \alpha)$



```
*****************************************************************
 EXAMPLE OF BETA VARIATIONS
*****************************************************************
```

Alpha = 2.0

VALUE OF DET. ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.773562	.000000	.773562
2	20.0	.796760	.141368	.809205
3	40.0	.886477	.300818	.936127
4	60.0	1.141333	.473801	1.235770
5	80.0	1.800178	.305875	1.825979
6	100.0	1.562908	-.754417	1.735461
7	120.0	.858667	-.636998	1.069147
8	140.0	.658329	-.364631	.752564
9	160.0	.599131	-.165345	.621528
10	180.0	.585118	.000000	.585118

Min value of Det. = .5851179957389831
Bound on det[I+K*PHI*B] : 6.499572522470694E-02

```
************************
BETA = 1.6
************************
```

VALUE OF DET. ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.559122	.000000	.559122
2	20.0	.557585	.176790	.584941
3	40.0	.547520	.406537	.681946
4	60.0	.482738	.825383	.956187
5	80.0	-.415468	2.076991	2.118137
6	100.0	-1.184669	-1.541263	1.943947
7	120.0	.103186	-.744650	.751765
8	140.0	.291298	-.380133	.478911
9	160.0	.346913	-.167374	.385179
10	180.0	.360760	.000000	.360760

Min value of Det. = .3607599735260009

****
BETA = 1.7
****

VALUE OF DET. ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.529736	.000000	.529736
2	20.0	.524422	.175425	.552985
3	40.0	.498587	.400480	.639510
4	60.0	.379265	.789046	.875463
5	80.0	-.612616	1.559769	1.675761
6	100.0	-1.160593	-.994149	1.528172
7	120.0	.021723	-.676334	.676683
8	140.0	.247874	-.362191	.438889
9	160.0	.316186	-.161630	.355102
10	180.0	.333199	.000000	.333199

Min value of Det. = .3331994414329528

It can be seen from the example presented above that for a value of β, such that $1 < \beta < \alpha$, the values of the determinant of the RDM of the system $(\beta A, \beta B, K_\alpha)$ satisfy the lower bound of Λ_α. Hence this system possesses the stability margins corresponding to Λ_α. Figure (3-f) illustrates the result graphically.

Due to perturbations in the system, in the worst case, the CL poles will be pushed nearer to the unit circle. A system designed with a prescribed degree of stability will have more capacity for tolerating perturbations. An explicit condition on the perturbations ΔA was not possible, as mentioned earlier. But a numerical example will be presented to justify the advantages of a system designed with prescribed degree of stability. The system considered is from [21].

System Order = 2
No. of Inputs = 1

System Matrix [A]

$$\begin{bmatrix} 0.0 & 1.0 \\ -0.5 & -1.0 \end{bmatrix}$$

Input Matrix [B]

$$\begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}$$

State weight Matrix [Q]

$$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Input weight Matrix [R]

1.0

Perturbation Matrix [D]

$$\begin{bmatrix} 0.0 & 0.0 \\ -0.9 & -0.9 \end{bmatrix}$$

Eigenvalues of A are

```
**************************************************
Alpha =  1.0
**************************************************
 Lower Bound on DET( I + K * PHI * B )
 [Equation(3.29)]
 0.5190440862106403
```

The solution to DARE is

$$\begin{bmatrix} 1.18264830834147 & .3219313728029613 \\ .3219313728029613 & 2.71185929880689 \end{bmatrix}$$

The LQR controller is

[-.365296616682948 -.643862745605923]

Closed loop matrix is

$$\begin{bmatrix} (.0,.0) & (1.0,.0) \\ (-.1347,.0) & (-.35614,.0) \end{bmatrix}$$

Magnitudes of Closed loop poles are
.367019
.367019

Closed loop perturbed matrix is

$$\begin{bmatrix} (.0,.0) & (1.0,.0) \\ (-1.0347,.0) & (-1.2561,.0) \end{bmatrix}$$

Mag.of closed loop poles of the perturbed system are
1.0172
1.0172

```
**************************************************
Alpha =  1.4
**************************************************
 Lower Bound on DET( I + K * PHI * B )
 [Equation(3.29)]
 0.3349261815598269
```

The solution to DARE is

$$\begin{bmatrix} 1.22195611315436 & .3999454128983464 \\ .3999454128983464 & 4.03806062354411 \end{bmatrix}$$

Eigen-values of alp^2 Palpha - P are
(1.1596,.0)
(5.2555,.0)

The LQR controller is

[-.443912226308728 -.799890825796692]

Closed loop matrix is

$$\begin{bmatrix} (.0,.0) & (1.0,.0) \\ (-5.60877E-02,.0) & (-.20011,.0) \end{bmatrix}$$

$$\begin{bmatrix} (.0,.0) & (1.0,.0) \\ (-.95609,.0) & (-1.1001,.0) \end{bmatrix}$$

Mag.of Closed loop poles of the perturbed system are
.977797
.977797

Alpha = 1.4

Bound on det[I+K*PHI*B] : 0.3349261815598269

VALUE OF DET. ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.411268	.000000	.411268
2	12.9	.409121	.058924	.413342
3	25.7	.402400	.120196	.419968
4	38.6	.390180	.186594	.432501
5	51.4	.370574	.261942	.453804
6	64.3	.339996	.352264	.489579
7	77.1	.291357	.468366	.551594
8	90.0	.209149	.632775	.666444
9	102.9	.052933	.903390	.904939
10	115.7	-.322642	1.493293	1.527750
11	128.6	-2.073016	4.400855	4.864660
12	141.4	4.255364	-3.390511	5.440927
13	154.3	1.896805	-.963033	2.127276
14	167.1	1.515123	-.360730	1.557473
15	180.0	1.430654	.000000	1.430654

Min value of Det. = .4112676978111267

****
For perturbed system
****

VALUE OF DET. ON |Z| = 1.0

Sno.	Th	Det		|Det|
1	.0	.689270	.000000	.689270
2	12.9	.687400	.011457	.687495
3	25.7	.681616	.022640	.681992
4	38.6	.671368	.033136	.672185
5	51.4	.655640	.042189	.656996
6	64.3	.632802	.048298	.634642
7	77.1	.600451	.048355	.602395
8	90.0	.555586	.035718	.556733
9	102.9	.496760	-.004046	.496777
10	115.7	.435104	-.100152	.446482
11	128.6	.432355	-.287868	.519422
12	141.4	.623193	-.496866	.797022
13	154.3	.957408	-.484108	1.072842
14	167.1	1.171339	-.262694	1.200435

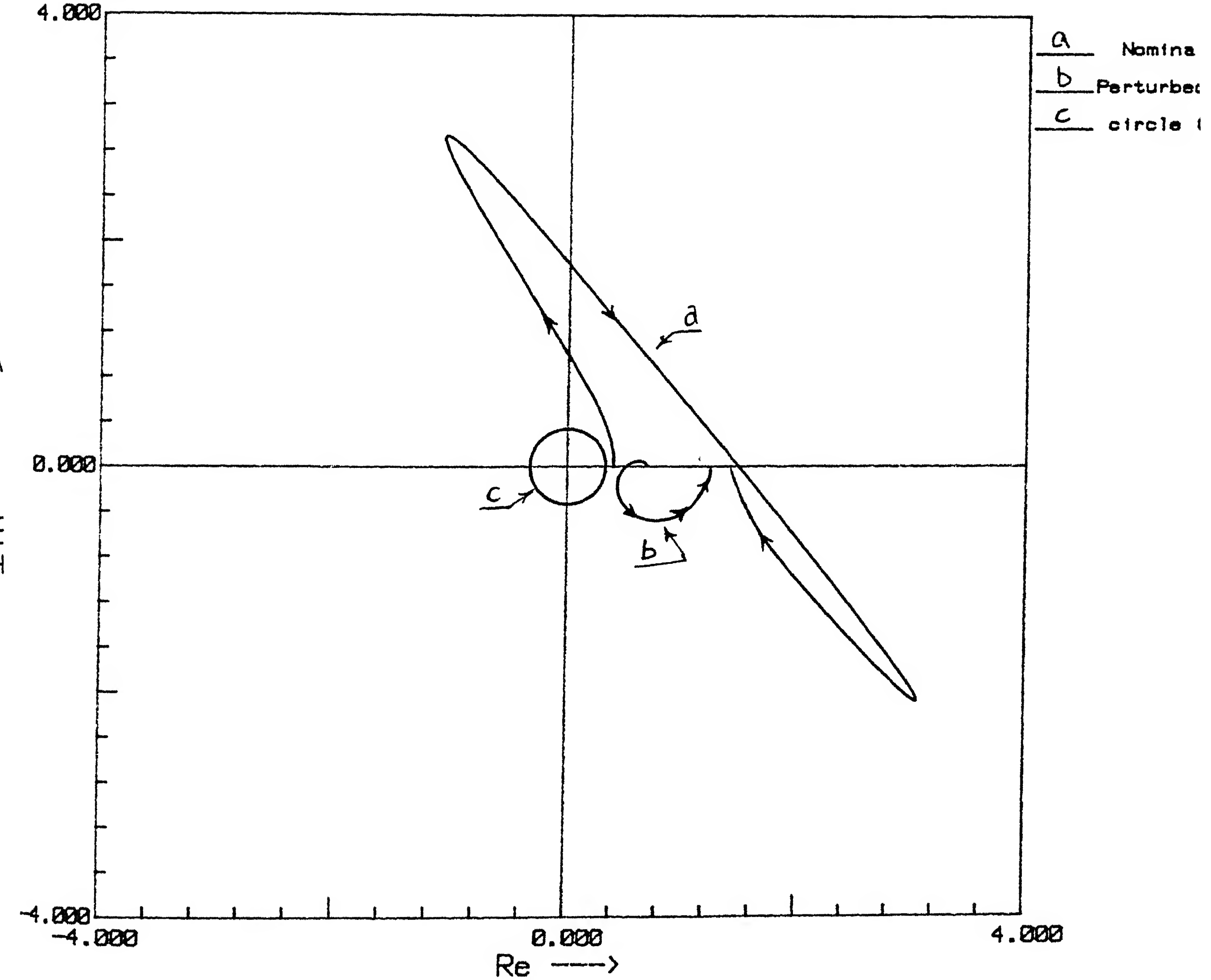


figure 3-g Det. of RDMfor pertuebed system

It can be seen from the numerical example presented that with the given perturbation, if an optimal controller without any prescribed degree of stability is used, then the perturbed system does not remain stable. It is found that for the stability of the perturbed system, an optimal controller with a minimum degree of stability $\alpha = 1.4$ is required. With $\alpha = 1.4$, the perturbed system not only remains stable but also satisfies the bound for optimality, resulting in guaranteed stability margins corresponding to the radius $\Lambda_{1.4} = 0.3349262$. Theoretically, for all values of $\alpha > 1.4$, the CL perturbed system becomes stable, but it is observed that a very high value of α makes such a design numerically impossible. A solution to the DARE in such case becomes very large and it creates problems of convergence.

3.4 Conclusions

The LQR with prescribed degree of stability inherently possesses greater tolerance to perturbations. Developing the return difference equality for the perturbed system is possible in case of CLQR, with some condition on the degree of stability. The structure of the controller in case of DLQR makes such a result impossible in case of discrete-time systems. A DLQR with prescribed degree of stability α, is shown to have guaranteed stability margins with respect to the circle of radius $1/\alpha$, centered at the origin. Systems with lesser degrees of stability satisfy the necessary condition for optimality for systems with higher degrees of stability. This property can be used in the design of a controller for systems with known perturbations.

CHAPTER # 4

CONCLUSIONS

Owing to certain differences between the continuous-time and discrete-time LQR, a condition similar to that derived in [6] for continuous-time systems was not possible for discrete-time systems.

The discrete-time LQR possesses some guaranteed stability margins, though these are small as compared to those for CLQR. It is shown that the DLQR with some prescribed degree of stability, α, ($\alpha > 1$) possesses guaranteed stability margins for the CL system with respect to a circle of radius $1/\alpha$, centered at the origin on the z-plane. i.e. These margins are the measures of the nearness of the CL poles of the system to the above mentioned circle of radius $1/\alpha$.

It is shown that if a system with 'β A' and 'β B' as its state and input matrices respectively, uses a controller K_α designed for the system (A,B), then the resulting CL system has the following property. Let us assume that the design model (A,B,K_α) has GM1,GM2 and PM as guaranteed stability margins (upward gain margin, downward gain margin and phase margin respectively) with respect to circle $|z| = 1/\alpha$ on the z-plane. Then the actual system $(\beta A,\beta B,K_\alpha)$ is guaranteed to have GM1, GM2 and PM as its margins of stability with respect to circle $|z| = 1$, as long as $1 \le \beta \le \alpha$.

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